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Lectures Notes, Math887
24apr03

Lecture 2: Poisson's summation formula

1. The formula

In essence, the PSF shows that two possibly different periodizations of a function/distribution are actually the same.

The simplest version of the PSF is as follows:

Theorem 1.1. *Let $f \in L_1(\mathbb{R})$. Let*

$$f^\circ := f*'1 := \sum_{j\in\mathbb{Z}} E^j f$$

be the **periodization** *of f. Then the Fourier series of f° is*

$$\sum_{\alpha\in 2\pi\mathbb{Z}} \widehat{f}(\alpha) e_{i\alpha}.$$

Proof: Since $f \in L_1(\mathbb{R})$, it is easy to see that the series $f*'1$ converge in $L_1([0\,..\,1])$ (prove it!). In order to prove the theorem, we just need to compute the Fourier coefficients of the resulting f°. However, for any bounded periodic function g, we have that

$$\int_{[0..1]} (f*'1)(t) g(t)\,\mathrm{d}t = \int_{\mathbb{R}} f(t) g(t)\,\mathrm{d}t$$

(prove it! in particular, justify the interchange of summation and integration here). Choosing $g := e_{-i\alpha}$, we obtain that the α-coefficient of $f*'1$ is $\widehat{f}(\alpha)$, which is exactly what is claimed. □

Another version of the PSF is that

$$\sum_{j\in\mathbb{Z}} f(j) = \sum_{\alpha\in 2\pi\mathbb{Z}} \widehat{f}(\alpha). \tag{1.2}$$

Note that this identity follows by an evaluation of the relation

$$f*'1 = \sum_{\alpha\in 2\pi\mathbb{Z}} \widehat{f}(\alpha) e_{i\alpha}$$

at the origin. However, one must be careful here: first, the Fourier series of $f*'1$ may not converge in a rigorous sense to that function; second, even if it does converge, the convergence may not be pointwise everywhere... However, it is easy to describe situations when (1.2) makes sense and is true (warning: there are notorious examples where both sides of (1.2) make sense but the equality there is invalid). A simple, yet useful, example is as follows:

Corollary 1.3. *Let f be a rapidly decaying continuous function, and assume that $(\widehat{f}(\alpha))_{\alpha\in 2\pi\mathbb{Z}}$ is in ℓ_1. Then (1.2) holds for this function.*

Proof: Due to the assumption here on $\widehat{f}$, the series $\sum_{\alpha\in 2\pi\mathbb{Z}} \widehat{f}(\alpha) e_{i\alpha}$ converges absolutely and uniformly to some function in $C([0\,..\,1])$. Theorem 1.1 identifies that function as $f*'1$, hence $(f*'1)(0) = \sum_{\alpha\in 2\pi\mathbb{Z}} \widehat{f}(\alpha)$. The equality $(f*'1)(0) = \sum_{j\in\mathbb{Z}} f(j)$ is justified by the fact that f is rapidly decaying and continuous. □

Another useful version of the PSF is the distributional version:

Corollary 1.4. *The identity*

$$\sum_{j\in\mathbb{Z}} \delta_j = \sum_{\alpha\in 2\pi\mathbb{Z}} e_{\mathrm{i}\alpha}$$

is valid in the distributional sense, as well as in the sense of tempered distributions.

Proof: Let f be a test function. Then

$$\langle \sum_j \delta_j, f\rangle = \sum_j f(j),$$

while

$$\langle \sum_\alpha e_{\mathrm{i}\alpha}, f\rangle = \sum_\alpha \widehat{f}(\alpha),$$

so the claim follows from the fact that test functions satisfy (1.2) (by Corollary 1.3). □

The final version of the PSF that we discuss is less well known. The following theorem states the 'compact support version' of it.

Theorem 1.5. *Let ϕ be a compactly supported distribution, and let f be in $C^\infty(\mathbb{R})$. Then*

$$\phi *' f = \sum_{\alpha\in 2\pi\mathbb{Z}} \phi *(e_{\mathrm{i}\alpha} f).$$

The convergence of both sides here holds in the topology of distributions. If f grows only slowly at ∞, that convergence holds also in the topology of tempered distributions.

Proof: Since f is infinitely differentiable, it is a multiplier in the space of distributions (with the additional assumption of slow growth, it is also a multiplier in the tempered distribution space). Thus we get from Corollary 1.4 that

$$\phi *(f \sum_j \delta_j) = \phi *(f \sum_\alpha e_{\mathrm{i}\alpha}).$$

Now, since convolution with a compactly supported function, as well as multiplication by a multiplier, are both continuous in the space of (tempered) distributions, and since the sum $\sum_\alpha e_{\mathrm{i}\alpha}$ converges in that space, we obtain

$$\phi *(f \sum_\alpha e_{\mathrm{i}\alpha}) = \phi *(\sum_\alpha f e_{\mathrm{i}\alpha}) = \sum_\alpha \phi *(f e_{\mathrm{i}\alpha}).$$

Similar reasoning shows that

$$\phi *(f \sum_j \delta_j) = \phi *(\sum_j f\delta_j) = \phi *(\sum_j f(j)\delta_j) = \sum_j f(j)\phi * \delta_j = \phi *' f.$$

□

2. Applications of the PSF

Lemma 2.1. *Let $(f_k)_k$ be a sequence of tempered distributions, and assume that $(\operatorname{supp} \widehat{f_k})_k$ are pairwise disjoint sets. Then $(f_k)_k$ is linearly independent in the sense that, if $\sum_k c(k) f_k$ converges to 0 (in the temp.dist. topology) for some coefficients $(c(k))_k$, then $c(k) = 0$, all k.*

Proof: The Fourier transform is continuous on the space of tempered distributions, hence, if $\sum_k c(k) f_k = 0$, then also $\sum_k c(k)\widehat{f_k} = 0$. Also, it is elementary to prove that any collection of distributions (tempered or not) with pairwise disjoint supports is linearly independent. □

Recall that $\mathcal{L} := 2\pi\mathbb{Z}\backslash 0$.

Corollary 2.2. *Let ϕ be compactly supported, and let p be some polynomial. If $\sum_{\alpha\in\mathcal{L}} \phi*(e_{\mathrm{i}\alpha}p)$ is a polynomial, then $\phi*(e_{\mathrm{i}\alpha}p) = 0$, for each $\alpha \in \mathcal{L}$.*

Proof: The Fourier transform of $\phi*(e_{\mathrm{i}\alpha}p)$ is supported at $\{-\alpha\}$ (prove it!), hence the Fourier transform of $\sum_{\alpha\in\mathcal{L}} \phi*(e_{\mathrm{i}\alpha}p)$ is supported on $\mathcal{L}$. Being assumed to be a polynomial, it must then be zero, since the Fourier transform of any polynomial is supported at $\{0\}$. □

Lemma 2.3. *Let p be a polynomial, $\alpha \in \mathbb{C}$, and ϕ compactly supported. Then:*

$$\phi*(e_{\mathrm{i}\alpha}p) = e_{\mathrm{i}\alpha} \sum_{j\ge 0} \frac{((-\mathrm{i}D)^j \widehat{\phi})(\alpha)}{j!} D^j p. \tag{2.4}$$

In particular (choose $\alpha = 0$), $\phi*(\Pi_k) \subset \Pi_k$ for all k; also (since $(D^j p : j = 0, \ldots, \deg p)$ is evidently linearly independent), $\phi*e_{\mathrm{i}\alpha}p = 0$ iff $\widehat{\phi}$ has a zero at α of order at least $1 + \deg p$.

Proof: It suffices to prove (2.4) for p an arbitrary monomial $()^m : x \mapsto x^m$. I will also assume for simplicity that ϕ is a function (and not merely a distribution). Then

$$\begin{aligned}
\phi*(e_{\mathrm{i}\alpha}()^m) &= \int_{\mathbb{R}} \phi(t)\mathrm{e}^{\mathrm{i}\alpha(\cdot - t)}(\cdot - t)^m \,\mathrm{d}t \\
&= e_{\mathrm{i}\alpha} \sum_{j=0}^{m} \binom{m}{j} ()^{m-j} \int_{\mathbb{R}} \phi(t)\mathrm{e}^{-\mathrm{i}\alpha t}(-t)^j \,\mathrm{d}t \\
&= e_{\mathrm{i}\alpha} \sum_{j=0}^{m} \frac{1}{j!} D^j(()^m)\,(-\mathrm{i}D)^j\Big(\int_{\mathbb{R}} \phi(t)\mathrm{e}^{-\mathrm{i}(\cdot)t}\,\mathrm{d}t\Big)(\alpha) \\
&= e_{\mathrm{i}\alpha} \sum_{j\ge 0} \frac{1}{j!} D^j(()^m)\,((-\mathrm{i}D)^j \widehat{\phi})(\alpha).
\end{aligned}$$

□

Theorem 2.5. *Let ϕ be a compactly supported distribution/function, and let k be a non-negative integer. Then $\phi*'$ maps $\Pi_{<k}$ into itself if and only if $\widehat{\phi}$ has a k-fold zero at each $\alpha \in \mathcal{L}$, and, in this case, $\phi*' = \phi*$ on $\Pi_{<k}$. Moreover, this map is onto $\Pi_{<k}$ if (in addition) $\widehat{\phi}(0) \ne 0$.*

Proof: "$\Longrightarrow$" Let $p \in \Pi$. By Theorem 1.5,

$$\phi*'p - \phi*p = \sum_{\alpha\in\mathcal{L}} \phi*(e_{\mathrm{i}\alpha}p).$$

Thus, if we assume that $\phi*'p$ is a polynomial, then, with Lemma 2.3, $\sum_{\alpha\in\mathcal{L}} \phi*(e_{\mathrm{i}\alpha}p)$ is a polynomial, hence (by Corollary 2.2) $\phi*(e_{\mathrm{i}\alpha}p) = 0$ for each $\alpha \in \mathcal{L}$. Invoking Lemma 2.3, we conclude that $\widehat{\phi}$ has a $(1 + \deg p)$-fold zero at each $\alpha \in \mathcal{L}$. Since, by assumption, $\phi*'$ maps $\Pi_{<k}$ into itself, we can now choose p to be some polynomial of degree $k - 1$, and we are done.

"$\Longleftarrow$" Assume that $\widehat{\phi}$ has a k-fold zero at each $\alpha \in \mathcal{L}$. Then, given $p \in \Pi_{<k}$, $\phi*(e_{\mathrm{i}\alpha}p) = 0$ for every $\alpha \in \mathcal{L}$ (Lemma 2.3), hence $\phi*'p = \phi*p$ (Theorem 1.5), while, by (2.4), $\phi*p \in \Pi_{<k}$.

Finally, (2.4) implies that

$$\phi*p = \widehat{\phi}(0)p + \text{l.o.t.}$$

(l.o.t. := 'lower order terms'). This shows that, if $\widehat{\phi}(0) \ne 0$, then $\phi*$ is injective on Π, hence in particular on $\Pi_{<k}$, hence $\phi*'$ is injective on that latter space, too. □

Theorem 2.6. *Let ϕ be a compactly supported distribution, and let $\vartheta \in \mathbb{C}$. Then $\widehat{\phi}$ vanishes on $\vartheta + 2\pi\mathbb{Z}$ iff $\phi *' e_{i\vartheta} = 0$.*

Proof: Assume first that $\vartheta = 0$, hence $e_{i\vartheta} = 1$. By Theorem 2.5, $\phi *' 1 = 0$ iff $\widehat{\phi}$ vanishes on $\mathcal{L}$ and (by (2.4)) $\widehat{\phi}(0) = 0$, i.e., iff $\widehat{\phi}$ vanishes on $2\pi\mathbb{Z}$.

If $\vartheta \neq 0$, then we can use the fact that $\phi *' e_{i\vartheta} = 0$ iff $(e_{-i\vartheta}\phi) *' 1 = 0$ (since always $e_\theta(f *' g) = (e_\theta f) *' (e_\theta g)$). Thus, $\phi *' e_{i\vartheta} = 0$ iff the Fourier transform of $e_{-i\vartheta}\phi$ vanishes on $2\pi\mathbb{Z}$. However, that transform is $\widehat{\phi}(\cdot + \vartheta)$. □