Lecture 3: Representation and bases in Hilbert spaces

Let $H$ be a separable Hilbert space and let $X \subset H$ be a countable subset that is fundamental, i.e., $\text{span}(X) = H$. One often wants to be able to express the elements $h$ of $H$ as linear combinations $h = \sum_{x \in X} c_h(x)x$ of elements of $X$. For this representation to be useful, there must be (1) a good way to compute the coefficients $(c_h(x))_{x \in X}$ form the vector $h$, (2) a good way to reconstruct $h$ from its coefficients $(c_h(x))_{x \in X}$, (3) the coefficients $(c_h(x))_{x \in X}$ must “tell the story”, i.e., reflect properties of $h$ so that one would gain by working with the coefficients rather than with the vector $h$ itself, and (4) $X$ cannot contain many essentially different elements (put positively, the elements in $X$ should be obtainable from a few “atoms” by simple operations like translation, multiplication by some functions, and dilation).

**Definition 1.** The linear map

$$T_X : \ell_2(X) \to H : c \mapsto \sum_{x \in X} c(x)x$$

is called the synthesis operator.

This definition may not make sense without further assumptions. Observe, however, that the action of $T_X$ on sequences with finite support, i.e., on the linear space

$$\ell_0(X) := \{c \in \Phi^X : \# \text{supp } c < \infty\},$$

is well defined, and $\ell_0(X)$ is dense in $\ell_2(X)$, hence $T_X$ is always densely defined. If, in addition, $\|T_X|_{\ell_0(X)}\| < \infty$, one can (uniquely) extend $T_X|_{\ell_0(X)}$ to all of $\ell_2(X)$, preserving its norm, and call this extension $T_X$. For this reason, one makes the following

**Definition 2.** $X$ is called a Bessel system if $T_X|_{\ell_0(X)}$ is bounded (and then $T_X$ is defined on all of $\ell_2(X)$).

**Example 3.** Let $\phi \in L_2(\mathbb{R})$, $X := E(\phi) := (E^j\phi)_{j \in \mathbb{Z}}$, and $H := S(\phi) := \text{span}(E(\phi))$ (here, of course, the closure is taken in the topology of $L_2(\mathbb{R})$).

**Exercise 1.** In Example 3, $E(\phi)$ is Bessel if and only if $[\hat{\phi}, \hat{\phi}] \in L_\infty(\mathbb{R})$. Moreover, $\|T_X\|^2 = \|[\hat{\phi}, \hat{\phi}]\|_{L_\infty(\mathbb{R})}$. Here,

$$[f, g] := \sum_{\alpha \in 2\pi \mathbb{Z}} E^\alpha f \overline{E^\alpha g} = \sum_{\alpha \in 2\pi \mathbb{Z}} E^\alpha(f \overline{g}) \quad \forall f, g \in L_2(\mathbb{R})$$

is the bracket product of $f$ and $g$, i.e., the $2\pi$-periodization of $f \overline{g}$.

Note that, if $\phi, \psi \in L_2(\mathbb{R})$, then $\hat{\phi}, \hat{\psi} \in L_2(\mathbb{R})$, hence $\hat{\phi}\hat{\psi} \in L_1(\mathbb{R})$, hence $[\hat{\phi}, \hat{\psi}]$ is the $L_1$-limit of the sum $\sum_{\alpha \in 2\pi \mathbb{Z}} E^\alpha(\hat{\phi}\hat{\psi})$, hence in $L_1(\mathbb{T})$. This is useful information since, at least for any finitely supported $c$, $(T_E(\phi)c)^\wedge = \sum_j c(j)(\hat{\phi}(\cdot - j))^\wedge = \sum_j c(j)e_{-ij}\hat{\phi}$, therefore

$$(T_E(\phi)c)^\wedge = \hat{\phi},$$

with the $2\pi$-periodic function

$$\hat{c} := \sum_j c(j)e_{-ij}$$
the discrete Fourier transform of the sequence $c$. Therefore, by the Plancherel identity and elementary properties of the Fourier transform, at least for any finitely supported $c$ and $d$, 

\begin{equation}
\langle T_{E(\phi)} c, T_{E(\psi)} d \rangle = \frac{1}{2\pi} \langle \hat{c} \phi, \hat{d} \psi \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{c} \hat{\phi} = \frac{1}{2\pi} \int_{\mathbb{T}} \overline{c} \hat{\phi} \psi \rangle.
\end{equation}

In particular, with $c = \delta_0$, $d = \delta_j$, we find that 

\begin{equation}
\langle \phi, \psi(-j) \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} c \hat{\phi} \psi = \langle \hat{\phi}, \hat{\psi} \rangle^{(-j)}.
\end{equation}

As a corollary to this formula, we have: $\langle \hat{\phi}, \hat{\psi} \rangle = 1$ a.e. iff $\langle \phi, E^j \psi \rangle = \delta_{j0}$.

**Exercise 1.1** Use Exercise 1 to prove that $E(\phi)$ is Bessel in case $\phi$ decays ‘mildly’ at $\infty$, and also in case $\phi$ is ‘mildly smooth’. Quantify, for each case, the notion of ‘mild decay’ and ‘mild smoothness’.

**Observation:** For any $R \in bL(H)$, $RT_X = T_{RX}$, since both sides are bounded linear maps that agree on the fundamental set $X$ (since, for any $x \in X$, $RT_X \delta_x = Rx = T_{RX} \delta_x$).

**Proposition 5.** Let $X$ be a Bessel system. Then TFAE:

1. $T_X$ is bounded below, i.e., $\inf_{c \neq 0} \|T_X c\|_H / \|c\|_{\ell_2(X)} > 0$.
2. ran $T_X$ is closed and $\ker T_X = \{0\}$.
3. There exists $R \in bL(H)$ s.t. $(Rx, x') = \delta_{xx'} \ \forall x, x' \in X$.

**Proof:** (1)$\implies$\quad (2) This equivalence holds for any bounded linear map $T$ from a Banach space to a Banach space.

(2)$\implies$\quad (3) Since ran $T_X = \overline{\text{ran} T_X} \supseteq \text{span}(X) = H$, $T_X$ is onto $H$. A 1-1 map from a Banach space onto a Banach space is boundedly invertible by the Open Mapping Theorem. So, there exists $T_X^{-1} \in bL(H, \ell_2(X))$.

Let $R := (T_X^{-1})^*T_X^{-1}$, where * denotes the adjoint of an operator. Then $R \in bL(H)$ and $\langle Rx, x' \rangle = \langle T_X^{-1}x, T_X^{-1}x' \rangle_{\ell_2(X)} = \langle \delta_x, \delta_{x'} \rangle_{\ell_2(X)} = \delta_{xx'}$.

(3)$\implies$\quad (1) The bi-orthogonality condition in (3) states that, for any $x, x' \in \ell_2(X)$, $\langle Rx, x' \rangle = \langle T_{RX} \delta_x, T_X \delta_{x'} \rangle = \langle \delta_x, T_{RX}^* T_X \delta_{x'} \rangle$, hence $T_{RX}^* T_X = \text{id}$ on $\ell_0(X)$. Since $T_X$ is bounded, so is $T_{RX} = RT_X$, hence so is $T_{RX}^*$. and so, the above identity extends to all of $\ell_2(X)$. This means that $T_X$ has a bounded left inverse, hence must be bounded below.

**Exercise 2.** The dual system $RX$ is unique (since $X$ is fundamental for $H$).

**Definition 6.** A Bessel set $X \subset H$ is called a Riesz basis (or a stable basis) if the conditions (1)–(3) above are satisfied.

**Definition 7.** The map $T_X^* : H \to \ell_2(X) : h \mapsto (\langle h, x \rangle : x \in X)$ is the **analysis operator**.

This definition may not make sense for an arbitrary $X \subset H$, since the sequences $(\langle h, x \rangle : x \in X)$ do not have to lie in $\ell_2(X)$. However,

**Exercise 3.** If $T_X \in bL(\ell_2(X), H)$ or $T_X^* \in bL(H, \ell_2(X))$, then both maps are bounded, have the same norm, and $T_X^*$ is the adjoint of $T_X$. Also (given that $X$ is fundamental), $T_X^*$ is 1-1.

The following is standard.

**Proposition 8.** Suppose $H_1$, $H_2$ are Hilbert spaces and $T \in bL(H_1, H_2)$. Then ran $T$ is closed in $H_2$ if and only if ran $T^*$ is closed in $H_1$. If this is the case, then $T$ is 1-1 iff $T^*$ is onto.

**Proposition 9.** Let $\phi \in L_2(\mathbb{R})$ be compactly supported, $X := E(\phi)$, and $H := S(\phi) \subseteq L_2(\mathbb{R})$. TFAE:

1. $E(\phi)$ is a Riesz basis for $H$.
2. $\hat{\phi}$ vanishes nowhere in $\mathbb{R}$.
(3) \( \hat{\phi} \) does not have a real \( 2\pi \)-periodic zero.

**Remark:** The assumption on \( \phi \) implies that \( \hat{\phi} \) is entire and, since \( \langle \hat{\phi}, e_{-ij} \hat{\phi} \rangle = [\hat{\phi}, \hat{\phi}]^\vee (j) = 0 \) for all but finitely many \( j \in \mathbb{Z} \), \([\hat{\phi}, \hat{\phi}]\) is equal a.e. to a trigonometric polynomial.

**Exercise 4.** Prove that \( \sum_{\alpha \in 2\pi \mathbb{Z}} E^\alpha [\hat{\phi}]^2 \) converges uniformly on compact sets, hence \( \hat{\phi}, \hat{\phi} \) is equal to that trigonometric polynomial everywhere. Prove the equivalence of (2) and (3).

**Remark:** With that, by (4), \( \| T_{E(\phi)} c \|^2 = \frac{1}{2\pi} \int_T |c(x)|^2 \| \hat{\phi} \|_2^2 \geq \inf \| \hat{\phi}, \hat{\phi} \|_2^2 \), for any finitely supported \( c \). Since \( \hat{\phi}, \hat{\phi} \) is continuous (and \( \ell_0 \) is dense in \( \ell_2 \)), this shows that (2)\( \implies \)(1), while if \( \hat{\phi}, \hat{\phi}(\theta) = 0 \), then \( \hat{\phi}, \hat{\phi} \) is \( O(\varepsilon) \) on \( B_r(\theta) \), hence \( \| T_{E(\phi)} c \|^2 \geq O(\varepsilon) \) with \( c \) chosen to have \( \| c \| \sim 1 \) yet \( \hat{c} \) small off \( B_r(\theta) \) (e.g., \( \hat{c} = D_{\{1/\varepsilon\}}(\cdot - \theta) \), the \( \theta \)-translate of a Dirichlet kernel), thus proving (1)\( \implies \)(2).

**Proof:** (1)\( \implies \)(2) Suppose that \( \hat{\phi}, \hat{\phi}(\theta) = 0 \) for some \( \theta \in \mathbb{R} \) (WLOG, \( \theta \in \mathbb{T} \), since \( \hat{\phi}, \hat{\phi} \) is \( 2\pi \)-periodic). For any \( \varepsilon > 0 \), define \( \chi_\varepsilon = \chi_{B_r(\theta)+2\varepsilon \mathbb{Z}} \), where \( B_r(x) \) denotes the ball of radius \( r \) centered at \( x \). Let \( c_\varepsilon = \frac{1}{2\pi} \langle (\chi_\varepsilon \mathbb{T})^\vee (a) \rangle_{a \in \mathbb{Z}} \). Then

\[
(10) \quad \| \chi_\varepsilon \mathbb{T} \|^2_{L_2(\mathbb{T})} = 2 \pi \| \chi_\varepsilon \|^2_{L_2(\mathbb{R})},
\]

On the other hand,

\[
(T_X c_\varepsilon)^\vee = \left( \sum_{j \in \mathbb{Z}} c_\varepsilon(j) e^{ij \phi} \right)^\vee = \sum_{j \in \mathbb{Z}} c_\varepsilon(j) e_{-ij} \hat{\phi} = \chi_\varepsilon \hat{\phi}
\]

since \( \sum_{j \in \mathbb{Z}} c_\varepsilon(j) e_{-ij} = \chi_\varepsilon \mathbb{T} \). Therefore,

\[
(11) \quad \| \hat{\phi} \|_{B_r(\theta)} \|_{L_1(\mathbb{T})} \leq \| \hat{\phi} \|_{B_r(\theta)} \|_{L_\infty(B_r(\theta))} \| \chi_\varepsilon \mathbb{T} \|_{L_1(\mathbb{T})} = \| \hat{\phi} \|_{B_r(\theta)} \|_{L_\infty(B_r(\theta))} \| \chi_\varepsilon \|_{L_2(\mathbb{T})}.
\]

Combining (10) and (11), we get

\[
\inf_{c \in L_2(\mathbb{R})} \frac{\| T_X c \|}{\| c \|} \leq \inf_{c \in L_\infty(B_r(\theta))} \frac{1}{\| \hat{\phi} \|_{B_r(\theta)}} = 0,
\]

since \( \hat{\phi}, \hat{\phi}(\theta) = 0 \) and \( \hat{\phi}, \hat{\phi} \) is continuous. But this means that condition (1) of Proposition 5 is violated, so \( X \) is not a Riesz basis.

(2)\( \implies \)(1) Since \( \hat{\phi}, \hat{\phi} \) is a continuous non-vanishing periodic function, \( \hat{\phi}, \hat{\phi}, -1 \in L_\infty(\mathbb{R}) \), hence \( \frac{\hat{\phi}}{\hat{\phi}, \hat{\phi}} \in L_2(\mathbb{R}) \) and \( \psi := \left( \frac{\hat{\phi}}{\hat{\phi}, \hat{\phi}} \right)^\vee \in L_2(\mathbb{R}) \). Note that \( \tau(f, g) = \tau(f, g) \) for any \( f, g \in L_2(\mathbb{R}) \) and a \( 2\pi \)-periodic function \( \tau \) s.t. \( \tau \in L_2(\mathbb{R}) \). Consequently,

\[
\hat{\psi}, \hat{\psi} = \left[ \frac{\hat{\phi}}{\hat{\phi}, \hat{\phi}}, \frac{\hat{\phi}}{\hat{\phi}, \hat{\phi}} \right] = \frac{\hat{\phi}, \hat{\phi}}{|\phi|^2} \frac{1}{|\phi, \phi|} \in L_\infty(\mathbb{R}),
\]

so, by Exercise 1 again, \( E(\psi) \) is Bessel. By Theorem 12 (to be proved in the nearest future), \( E(\psi) \subseteq H \). Let \( R : E(\phi) \rightarrow H : E^\alpha \phi \rightarrow E^\alpha \psi \). Show that \( RX = E(\psi) \) is the dual system for \( X \). Indeed, \( \hat{\psi}, \hat{\psi} = [\hat{\phi}, \hat{\phi}, \hat{\phi}] = 1 \), so, by the remark preceding Proposition 5, \( \langle \phi, E^\alpha \psi \rangle = \delta_{\alpha 0} \), hence \( \langle E^\alpha \phi, E^\beta \psi \rangle = \delta_{\alpha \beta} \| \hat{\phi}, \hat{\phi} \| \). So, the operator \( R \) extends to a map in \( bL(H) \) satisfying condition (3) of Proposition 5. Therefore, \( X \) is a Riesz basis.

**Theorem 12.** Let \( \phi, f \in L_2(\mathbb{R}) \). Then \( f \in S(\phi) \iff \hat{f} = \tau \hat{\phi} \) for some \( 2\pi \)-periodic function \( \tau \).