Amos Ron
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Prepared by Olga Holtz (then messed up by Carl de Boor)

Lecture 3: Representation and bases in Hilbert spaces

Let H be a separable Hilbert space and let $X \subset H$ be a countable subset that is **fundamental**, i.e., $\overline{\mathrm{span}(X)} = H$. One often wants to be able to express the elements h of H as linear combinations $h = \sum_{x\in X} c_h(x)x$ of elements of X. For this representation to be useful, there must be (1) a good way to compute the coefficients $(c_h(x))_{x\in X}$ form the vector h, (2) a good way to reconstruct h from its coefficients $(c_h(x))_{x\in X}$, (3) the coefficients $(c_h(x))_{x\in X}$ must "tell the story", i.e., reflect properties of h so that one would gain by working with the coefficients rather than with the vector h itself, and (4) X cannot contain many essentially different elements (put positively, the elements in X should be obtainable from a few "atoms" by simple operations like translation, multiplication by some functions, and dilation).

Definition 1. *The linear map*

$$T_X : \ell_2(X) \to H : c \mapsto \sum_{x\in X} c(x)x$$

is called the **synthesis operator**.

This definition may not make sense without further assumptions. Observe, however, that the action of T_X on sequences with finite support, i.e., on the linear space

$$\ell_0(X) := \{c \in \mathbb{C}^X : \#\,\mathrm{supp}\, c < \infty\},$$

is well defined, and $\ell_0(X)$ is dense in $\ell_2(X)$, hence T_X is always densely defined. If, in addition, $\|T_X|_{\ell_0(X)}\| < \infty$, one can (uniquely) extend $T_X|_{\ell_0(X)}$ to all of $\ell_2(X)$, preserving its norm, and call this extension T_X. For this reason, one makes the following

Definition 2. *X is called a Bessel system if $T_X|_{\ell_0(X)}$ is bounded (and then T_X is defined on all of $\ell_2(X)$).*

Example 3. Let $\phi \in L_2(\mathbb{R})$, $X := E(\phi) := (E^j\phi)_{j\in\mathbb{Z}}$, and $H := S(\phi) := \overline{\mathrm{span}(E(\phi))}$ (here, of course, the closure is taken in the topology of $L_2(\mathbb{R})$).

Exercise 1. In Example 3, $E(\phi)$ is Bessel if and only if $[\widehat{\phi},\widehat{\phi}] \in L_\infty(\mathbb{R})$. Moreover, $\|T_X\|^2 = \|[\widehat{\phi},\widehat{\phi}]\|_{L_\infty(\mathbb{R})}$. Here,

$$[f,g] := \sum_{\alpha\in 2\pi\mathbb{Z}} E^\alpha f\,\overline{E^\alpha g} = \sum_{\alpha\in 2\pi\mathbb{Z}} E^\alpha(f\overline{g}) \qquad \forall f,g \in L_2(\mathbb{R})$$

is the **bracket product** of f and g, i.e., the 2π-periodization of $f\overline{g}$.

Note that, if ϕ, $\psi \in L_2(\mathbb{R})$, then $\widehat{\phi}$, $\widehat{\psi} \in L_2(\mathbb{R})$, hence $\widehat{\phi}\overline{\widehat{\psi}} \in L_1(\mathbb{R})$, hence $[\widehat{\phi},\widehat{\psi}]$ is the L_1-limit of the sum $\sum_{\alpha\in 2\pi\mathbb{Z}} E^\alpha(\widehat{\phi}\overline{\widehat{\psi}})$, hence in $L_1(\mathbb{T})$. This is useful information since, at least for any finitely supported c, $(T_{E(\phi)}c)^\wedge = \sum_j c(j)(\phi(\cdot - j))^\wedge = \sum_j c(j)e_{-\mathrm{i}j}\widehat{\phi}$, therefore

$$(T_{E(\phi)}c)^\wedge = \widehat{c}\widehat{\phi},$$

with the 2π-periodic function

$$\widehat{c} := \sum_j c(j)e_{-\mathrm{i}j}$$

the discrete Fourier transform of the sequence c. Therefore, by the Plancherel identity and elementary properties of the Fourier transform, at least for any finitely supported c and d,

$$\langle T_{E(\phi)}c, T_{E(\psi)}d\rangle = \frac{1}{2\pi}\langle \widehat{c\phi}, \widehat{d\psi}\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{c}\overline{\widehat{d}}\widehat{\phi}\overline{\widehat{\psi}} = \frac{1}{2\pi}\int_{\mathbb{T}} \widehat{c}\overline{\widehat{d}}[\widehat{\phi}, \widehat{\psi}]. \tag{4}$$

In particular, with $c = \delta_0$, $d = \delta_j$, we find that

$$\langle \phi, \psi(\cdot - j)\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} e_{ij}[\widehat{\phi}, \widehat{\psi}] = [\widehat{\phi}, \widehat{\psi}]^\wedge(-j).$$

As a corollary to this formula, we have: $[\widehat{\phi}, \widehat{\psi}] = 1$ a.e. iff $\langle \phi, E^j\psi\rangle = \delta_{j0}$.

Exercise 1.1 Use Exercise 1 to prove that $E(\phi)$ is Bessel in case ϕ decays 'mildly' at ∞, and also in case ϕ is 'mildly smooth'. Quantify, for each case, the notion of 'mild decay' and 'mild smoothness'.

Observation: For any $R \in bL(H)$, $RT_X = T_{RX}$, since both sides are bounded linear maps that agree on the fundamental set X (since, for any $x \in X$, $RT_X\delta_x = Rx = T_{RX}\delta_x$).

Proposition 5. *Let X be a Bessel system. Then TFAE:*
(1) T_X is bounded below, i.e., $\inf_{c\neq 0} \|T_X c\|_H/\|c\|_{\ell_2(X)} > 0$.
(2) $\operatorname{ran} T_X$ is closed and $\ker T_X = \{0\}$.
(3) There exists $R \in bL(H)$ s.t. $\langle Rx, x'\rangle = \delta_{xx'}$ $\forall x, x' \in X$.

Proof: (1)$\Longleftrightarrow$(2) This equivalence holds for any bounded linear map T from a Banach space to a Banach space.

(2)$\Longrightarrow$(3) Since $\operatorname{ran} T_X = \overline{\operatorname{ran} T_X} \supseteq \overline{\operatorname{span}(X)} = H$, T_X is onto H. A 1-1 map from a Banach space onto a Banach space is boundedly invertible by the Open Mapping Theorem. So, there exists $T_X^{-1} \in bL(H, \ell_2(X))$. Let $R := (T_X^{-1})^* T_X^{-1}$, where $*$ denotes the adjoint of an operator. Then $R \in bL(H)$ and $\langle Rx, x'\rangle_H = \langle (T_X^{-1})^* T_X^{-1}x, x'\rangle_H = \langle T_X^{-1}x, T_X^{-1}x'\rangle_{\ell_2(X)} = \langle \delta_x, \delta_{x'}\rangle_{\ell_2(X)} = \delta_{xx'}$.

(3)$\Longrightarrow$(1) The bi-orthogonality condition in (3) states that, for any $x, x' \in$, $\delta_{xx'} = \langle Rx, x'\rangle = \langle T_{RX}\delta_x, T_X\delta_{x'}\rangle = \langle \delta_x, T_{RX}^* T_X \delta_{x'}\rangle$, hence

$$T_{RX}^* T_X = \mathrm{id}$$

on $\ell_0(X)$. Since T_X is bounded, so is $T_{RX} = RT_X$, hence so is T_{RX}^*, and so, the above identity extends to all of $\ell_2(X)$. This means that T_X has a bounded left inverse, hence must be bounded below. □

Exercise 2. The **dual system** RX is unique (since X is fundamental for H).

Definition 6. *A Bessel set $X \subset H$ is called a Riesz basis (or a stable basis) if the conditions (1)–(3) above are satisfied.*

Definition 7. *The map $T_X^* : H \to \ell_2(X) : h \mapsto (\langle h, x\rangle : x \in X)$ is the* **analysis operator**.

This definition may not make sense for an arbitrary $X \subset H$, since the sequences $(\langle h, x\rangle : x \in X)$ do not have to lie in $\ell_2(X)$. However,

Exercise 3. If $T_X \in bL(\ell_2(X), H)$ or $T_X^* \in bL(H, \ell_2(X))$, then both maps are bounded, have the same norm, and T_X^* is the adjoint of T_X. Also (given that X is fundamental), T_X^* is 1-1.

The following is standard.

Proposition 8. *Suppose H_1, H_2 are Hilbert spaces and $T \in bL(H_1, H_2)$. Then $\operatorname{ran} T$ is closed in H_2 if and only if $\operatorname{ran} T^*$ is closed in H_1. If this is the case, then T is 1-1 iff T^* is onto.*

Proposition 9. *Let $\phi \in L_2(\mathbb{R})$ be compactly supported, $X := E(\phi)$, and $H := S(\phi) \subseteq L_2(\mathbb{R})$. TFAE:*
(1) $E(\phi)$ is a Riesz basis for H.
(2) $[\widehat{\phi}, \widehat{\phi}]$ vanishes nowhere in $\mathbb{R}$.

(3) $\widehat{\phi}$ does not have a real 2π-periodic zero.

Remark: The assumption on ϕ implies that $\widehat{\phi}$ is entire and, since $\langle\widehat{\phi}, e_{-\mathrm{i}j}\widehat{\phi}\rangle = [\widehat{\phi},\widehat{\phi}]^\vee(j) = 0$ for all but finitely many $j \in \mathbb{Z}$, $[\widehat{\phi},\widehat{\phi}]$ is equal a.e. to a trigonometric polynomial.

Exercise 4. Prove that $\sum_{\alpha\in 2\pi\mathbb{Z}} E^\alpha|\widehat{\phi}|^2$ converges uniformly on compact sets, hence $[\widehat{\phi},\widehat{\phi}]$ is equal to that trigonometric polynomial *everywhere*. Prove the equivalence of (2) and (3).

Remark: With that, by (4), $\|T_{E(\phi)}c\|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |\widehat{c}|^2[\widehat{\phi},\widehat{\phi}] \geq \inf[\widehat{\phi},\widehat{\phi}]\|c\|^2$, for any finitely supported c. Since $[\widehat{\phi},\widehat{\phi}]$ is continuous (and ℓ_0 is dense in ℓ_2), this shows that (2)$\Longrightarrow$(1), while if $[\widehat{\phi},\widehat{\phi}](\theta) = 0$, then $[\widehat{\phi},\widehat{\phi}] = O(\varepsilon)$ on $B_\varepsilon(\theta)$, hence $\|T_{E(\phi)}c_\varepsilon\|^2 = O(\varepsilon)$ with c_ε chosen to have $\|c_\varepsilon\| \sim 1$ yet $\widehat{c_\varepsilon}$ small off $B_\varepsilon(\theta)$ (e.g., $\widehat{c_\varepsilon} = D_{\lceil 1/\varepsilon\rceil}(\cdot - \theta)$, the θ-translate of a Dirichlet kernel), thus proving (1)$\Longrightarrow$(2).

Proof: (1)$\Longrightarrow$(2) Suppose that $[\widehat{\phi},\widehat{\phi}](\theta) = 0$ for some $\theta \in \mathbb{R}$ (WLOG, $\theta \in \mathbb{T}$, since $[\widehat{\phi},\widehat{\phi}]$ is 2π-periodic). For any $\varepsilon > 0$, define $\chi_\varepsilon := \chi_{B_\varepsilon(\theta)+2\pi\mathbb{Z}}$, where $B_r(x)$ denotes the ball of radius r centered at x. Let $c_\varepsilon := \frac{1}{2\pi}\left((\chi_\varepsilon|_{\mathbb{T}})^\wedge(\alpha)\right)_{\alpha\in\mathbb{Z}}$. Then

$$\|\chi_\varepsilon|_{\mathbb{T}}\|^2_{L_2(\mathbb{T})} = 2\pi\|c_\varepsilon\|^2_{\ell_2(X)}. \tag{10}$$

On the other hand,

$$(T_X c_\varepsilon)^\wedge = \left(\sum_{j\in\mathbb{Z}} c_\varepsilon(j)E^j\phi\right)^\wedge = \sum_{j\in\mathbb{Z}} c_\varepsilon(j)e_{-\mathrm{i}j}\widehat{\phi} = \chi_\varepsilon\widehat{\phi}$$

since $\sum_{j\in\mathbb{Z}} c_\varepsilon(j)e_{-\mathrm{i}j}|_{\mathbb{T}} = \chi_\varepsilon|_{\mathbb{T}}$. Therefore,

$$\begin{aligned} 2\pi\|T_X(c_\varepsilon)\|^2_{L_2(\mathbb{R})} = \|(T_Xc_\varepsilon)^\wedge\|^2_{L_2(\mathbb{R})} = \|[\chi_\varepsilon\widehat{\phi},\chi_\varepsilon\widehat{\phi}]|_{\mathbb{T}}\|_{L_1(\mathbb{T})} = \|(\chi_\varepsilon[\widehat{\phi},\widehat{\phi}])|_{\mathbb{T}}\|_{L_1(\mathbb{T})} \\ \|[\widehat{\phi},\widehat{\phi}]|_{B_\varepsilon(\theta)}\|_{L_1(\mathbb{T})} \leq \|[\widehat{\phi},\widehat{\phi}]|_{B_\varepsilon(\theta)}\|_{L_\infty(B_\varepsilon(\theta))}\|\chi_\varepsilon|_{\mathbb{T}}\|_{L_1(\mathbb{T})} = \|[\widehat{\phi},\widehat{\phi}]|_{B_\varepsilon(\theta)}\|_{L_\infty(B_\varepsilon(\theta))}\|\chi_\varepsilon|_{\mathbb{T}}\|^2_{L_2(\mathbb{T})}. \end{aligned} \tag{11}$$

Combining (10) and (11), we get

$$\inf_{c\in\ell_2(X)} \frac{\|T_Xc\|}{\|c\|} \leq \inf_{\varepsilon>0} \|[\widehat{\phi},\widehat{\phi}]|_{B_\varepsilon(\theta)}\|^{1/2}_{L_\infty(B_\varepsilon(\theta))} = 0,$$

since $[\widehat{\phi},\widehat{\phi}](\theta) = 0$ and $[\widehat{\phi},\widehat{\phi}]$ is continuous. But this means that condition (1) of Proposition 5 is violated, so X is not a Riesz basis.

(2)$\Longrightarrow$(1) Since $[\widehat{\phi},\widehat{\phi}]$ is a continuous non-vanishing periodic function, $[\widehat{\phi},\widehat{\phi}]^{-1} \in L_\infty(\mathbb{R})$, hence $\frac{\widehat{\phi}}{[\widehat{\phi},\widehat{\phi}]} \in L_2(\mathbb{R})$ and $\psi := \left(\frac{\widehat{\phi}}{[\widehat{\phi},\widehat{\phi}]}\right)^\vee \in L_2(\mathbb{R})$. Note that $[\tau f, g] = \tau[f,g]$ for any f, $g \in L_2(\mathbb{R})$ and a 2π-periodic function τ s.t. $\tau f \in L_2(\mathbb{R})$. Consequently,

$$[\widehat{\psi},\widehat{\psi}] = [\frac{\widehat{\phi}}{[\widehat{\phi},\widehat{\phi}]}, \frac{\widehat{\phi}}{[\widehat{\phi},\widehat{\phi}]}] = \frac{[\widehat{\phi},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]^2} = \frac{1}{[\widehat{\phi},\widehat{\phi}]} \in L_\infty(\mathbb{R}),$$

so, by Exercise 1 again, $E(\psi)$ is Bessel. By Theorem 12 (to be proved in the nearest future), $E(\psi) \subseteq H$. Let $R : E(\phi) \to H : E^\alpha\phi \mapsto E^\alpha\psi$. Show that $RX = E(\psi)$ is the dual system for X. Indeed, $[\widehat{\phi},\widehat{\psi}] = [\widehat{\phi}, \frac{\widehat{\phi}}{[\widehat{\phi},\widehat{\phi}]}] = 1$, so, by the remark preceding Proposition 5, $\langle\phi, E^\alpha\psi\rangle = \delta_{\alpha 0}$, hence $\langle E^\alpha\phi, E^\beta\psi\rangle = \delta_{(\alpha-\beta)0} = \delta_{\alpha\beta}$. So, the operator R extends to a map in $bL(H)$ satisfying condition (3) of Proposition 5. Therefore, X is a Riesz basis. $\square$

Theorem 12. *Let ϕ, $f \in L_2(\mathbb{R})$. Then $f \in S(\phi)$ iff $\widehat{f} = \tau\widehat{\phi}$ for some 2π-periodic function τ.*