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Lectures Notes, Math887
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Lecture 4: Frames

If X is a Riesz basis, then T_X is boundedly invertible, hence so is its adjoint, and this implies that $T_X T_X^*$ is invertible. Setting, as we did in the previous lecture,

$$R := (T_X T_X^*)^{-1},$$

and noticing that R is self-adjoint, hence $T_X^* R = (RT_X)^* = T_{RX}^*$, we see that $\mathrm{id}_H = T_X T_X^* R = T_X T_{RX}^*$ and so obtain the expansion

$$h = T_X T_{RX}^* h = \sum_{x \in X} \langle h, Rx \rangle x \quad \forall h \in H.$$

So, provided we know that $T_X T_X^*$ is boundedly invertible and have in hand its inverse, R, we have a way to represent any vector h in H as a linear combination of vectors in X with easily computable coefficients. This gives a useful generalization of the notion of a Riesz basis.

Theorem 1. *Let H be a Hilbert space and let $X \subset H$ be a Bessel system for H. Then TFAE*
(1) T_X^* is bounded below.
(2) $\operatorname{ran} T_X^*$ is closed.
(3) There exists a Hilbert space $H' \supseteq H$ and a map $R : X \to H'$ s.t.
(i) $T_{RX} : c \mapsto \sum_{x \in X} c(x)Rx$ is bounded (on $\ell_0(X)$ hence on $\ell_2(X)$);
(ii) $T_X T_{RX}^* = \mathrm{id}$ on H.

Proof: First note that $\ker T_X^* = \{0\}$ since X is fundamental in H (that is hidden in the definition of 'a Bessel system for H'). Further, it is true of any two Banach spaces X_1, X_2 that the conditions (i) "A is bounded below" and (ii) "$\ker A = \{0\}$ and $\operatorname{ran} A$ is closed" are equivalent for a map $A \in bL(X_1, X_2)$. (We already used this in Proposition 5 of Lecture 3.) So, since $\ker T_X^*$ is trivial, and both $\ell_2(X)$ and H are Banach spaces, (1) and (2) are equivalent.

(1)$\Longrightarrow$(3) We only need to show that $T_X T_X^*$ is boundedly invertible since then, by the above discussion, $R = (T_X T_X^*)^{-1}$ does the job. In particular, R being bounded, so is $T_{RX} = RT_X$, i.e., RX is Bessel. For that, by assumption, $K := \inf_h \|T_X^* h\|/\|h\|$ is positive. Since

$$\|T_X T_X^* h\| \geq \frac{|\langle T_X T_X^* h, h \rangle|}{\|h\|} = \frac{\|T_X^* h\|^2}{\|h\|} \geq K^2 \|h\| \quad \forall h \in H,$$

also the map $T_X T_X^*$ is bounded below, hence its kernel is trivial and its range is closed. But since it is self-adjoint, the closure of its range is the orthocomplement of its kernel, hence, all of H. In other words, $T_X T_X^*$ is boundedly invertible.

(3)$\Longrightarrow$(1) Since $T_{RX} T_X^* = (T_X T_{RX}^*)^* = \mathrm{id}^* = \mathrm{id}$, the map T_X^* is bounded below (with $K = 1/\|T_{RX}\| > 0$ by (i)). □

Definition 2. *A Bessel system $X \subset H$ is called a* **frame** *if the conditions (1)-(3) of Theorem 1 above are satisfied.*

Remark: The dual system RX constructed in the above proof is the unique system in H for which the projector $T^*_{RX}T_X = T^*_X R^* T_X = T^*_X(T_X T^*_X)^{-1}T_X$ is self-adjoint, hence orthogonal, hence the orthoprojector onto $\operatorname{ran} T^*_X$, and its kernel is $\ker T_X$. However, unless X is a Riesz basis, there are other dual frames in H for X. Note that we could have introduced a super-space H' in Proposition 5 of Lecture 3 without changing the conclusion of that proposition or its proof.

Exercise. Let R be the map constructed in the proof of Theorem 1. Given $h \in H$, prove that $T^*_{RX}h$ has the least norm among all representation of h as a combination of X.

Example 1. Let $X := \{x_1, \ldots, x_n\} \subset \mathbb{R}^m$, $m \le n$. Then $T_X = [x_1, \ldots, x_n] \in \mathbb{R}^{m\times n}$ is a frame iff $T_X T' = I_{m\times m}$ for some $T' \in \mathbb{R}^{n\times m}$ iff the rows of the map T_X are linearly independent iff $\operatorname{rank} T_X = m$.

Theorem 3. *Let $\phi \in L_2(\mathbb{R})$. The system $E(\phi)$ is a frame (for $S(\phi)$) iff*
(i) $[\widehat{\phi}, \widehat{\phi}] \in L_\infty(\mathbb{R})$*; and*
(ii) $[\widehat{\phi}, \widehat{\phi}]^{-1} \in L_\infty(\Omega_\phi)$ *with* $\Omega_\phi := \operatorname{supp}[\widehat{\phi}, \widehat{\phi}]$.

Proof: First note that

$$(T_{E(\phi)}T^*_{E(\psi)}f)^\wedge = [\widehat{f}, \widehat{\psi}]\widehat{\phi} \quad \forall \phi, \psi, f \in L_2(\mathbb{R}) \quad \text{s.t. } [\widehat{f}, \widehat{\psi}]\widehat{\phi} \in L_2(\mathbb{R}). \tag{4}$$

Indeed,

$$\left(T_{E(\phi)}T^*_{E(\psi)}f\right)^\wedge = \left(\sum_{j\in\mathbb{Z}} \langle f, E^j\psi\rangle E^j\phi\right)^\wedge = \sum_{j\in\mathbb{Z}} \langle f, E^j\psi\rangle e_{-\mathrm{i}j}\widehat{\phi} = [\widehat{f}, \widehat{\psi}]\widehat{\phi},$$

since $\sum_{j\in\mathbb{Z}}\langle f, E^j\psi\rangle e_{-\mathrm{i}j}$ is the Fourier series of $[\widehat{f}, \widehat{\psi}]$ (cf. Lecture 3).

$\Longrightarrow$ If $E(\phi)$ is a frame, then, in particular, it is Bessel, so $[\widehat{\phi}, \widehat{\phi}] \in L_\infty(\mathbb{R})$ by Exercise 1 of Lecture 3. By Theorem 12 of Lecture 3, for any $f \in S(\phi)$, $\widehat{f} = \tau\widehat{\phi}$ for some 2π-periodic τ, hence, from (4),

$$\|T^*_{E(\phi)}f\|^2 = \langle T_{E(\phi)}T^*_{E(\phi)}f, f\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} [\widehat{f}, \widehat{\phi}]\widehat{\phi}\overline{\widehat{f}} = \frac{1}{2\pi}\int_{\mathbb{T}} |[\widehat{f}, \widehat{\phi}]|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |\tau[\widehat{\phi}, \widehat{\phi}]|^2.$$

For $\varepsilon > 0$, let $\Omega_\varepsilon := \{\xi \in \mathbb{T} : [\widehat{\phi}, \widehat{\phi}](\xi) \le \varepsilon\}$. By Theorem 12 of Lecture 3, f given implicitly by $\widehat{f} = \chi_{\Omega_\varepsilon + 2\pi\mathbb{Z}}\widehat{\phi}$ is in $S(\phi)$. Also, $\|f\|^2 = \frac{1}{2\pi}\|\widehat{f}\|^2 = \frac{1}{2\pi}\int_{\Omega_\varepsilon}[\widehat{\phi}, \widehat{\phi}]$, while $\|T^*_{E(\phi)}f\|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |[\widehat{f}, \widehat{\phi}]|^2 = \frac{1}{2\pi}\int_{\Omega_\varepsilon} |[\widehat{\phi}, \widehat{\phi}]|^2$, hence $\|T^*_{E(\phi)}f\|^2 \le \varepsilon\|f\|^2$. Since $T^*_{E(\phi)}$ is bounded below, this implies that $\operatorname{meas}\Omega_\varepsilon = 0$ for some $\varepsilon > 0$.

$\Longleftarrow$ Suppose the conditions (i) and (ii) hold. Define a function ψ by

$$\widehat{\psi}(\omega) = \begin{cases} \frac{\widehat{\phi}}{[\widehat{\phi}, \widehat{\phi}]}(\omega) & \text{if } \omega \in \Omega_\phi \\ 0 & \text{otherwise.} \end{cases}$$

This definition makes sense since, by (ii), the right hand side is in $L_2(\mathbb{R})$, so its inverse Fourier transform ψ is defined and also lies in $L_2(\mathbb{R})$. Note also that, by Theorem 12 of Lecture 3, $\psi \in S(\phi)$. Let $f \in S(\phi)$. By that theorem, $\widehat{f} = \tau\widehat{\phi}$ for some 2π-periodic τ, hence (4) implies

$$(T_{E(\phi)}T^*_{E(\psi)}f)^\wedge = [\widehat{f}, \widehat{\psi}]\widehat{\phi} = [\tau\widehat{\phi}, \frac{\widehat{\phi}}{[\widehat{\phi}, \widehat{\phi}]}]\widehat{\phi} = \frac{\tau}{[\widehat{\phi}, \widehat{\phi}]}[\widehat{\phi}, \widehat{\phi}]\widehat{\phi} = \tau\widehat{\phi} = \widehat{f}.$$

Thus, $T_{E(\phi)}T^*_{E(\psi)} = \mathrm{id}$ on $S(\phi)$; by Theorem 1, $E(\phi)$ is a frame.

Example.

Let U be a measurable subset of $\mathbb{T}$ and let $\phi := \chi_U^\vee$. Then $[\widehat{\phi}, \widehat{\phi}] = \sum_{j\in\mathbb{Z}} \chi_U(\cdot - 2\pi j) \in L_\infty(\mathbb{R})$ and $[\widehat{\phi}, \widehat{\phi}]^{-1} \in L_\infty(U + 2\pi\mathbb{Z})$, so $E(\phi)$ is a frame. However, $E(\phi)$ is a Riesz basis only if $U = \mathbb{T}$, according to Proposition 8, part 2, of Lecture 3.