Lecture 4: Frames

If $X$ is a Riesz basis, then $T_X$ is boundedly invertible, hence so is its adjoint, and this implies that $T_X^*T_X$ is invertible. Setting, as we did in the previous lecture,

$$R := (T_X^*T_X)^{-1},$$

and noticing that $R$ is self-adjoint, hence $T_X^*R = (RT_X)^* = T_{RX}^*$, we see that $\text{id}_H = T_XT_X^*R = T_XT_{RX}^*$ and so obtain the expansion

$$h = T_XT_{RX}^*h = \sum_{x \in X} \langle h, Rx \rangle x \quad \forall h \in H.$$

So, provided we know that $T_XT_X^*$ is boundedly invertible and have in hand its inverse, $R$, we have a way to represent any vector $h$ in $H$ as a linear combination of vectors in $X$ with easily computable coefficients. This gives a useful generalization of the notion of a Riesz basis.

**Theorem 1.** Let $H$ be a Hilbert space and let $X \subset H$ be a Bessel system for $H$. Then TFAE

1. $T_X^*$ is bounded below.
2. $\text{ran} \ T_X$ is closed.
3. There exists a Hilbert space $H' \supseteq H$ and a map $R : X \to H'$ s.t.
   
   i. $T_{RX} : c \mapsto \sum_{x \in X} c(x)Rx$ is bounded (on $\ell_0(X)$ hence on $\ell_2(X)$);
   
   ii. $T_XT_{RX}^* = \text{id}$ on $H$.

**Proof:** First note that $\ker T_X^* = \{0\}$ since $X$ is fundamental in $H$ (that is hidden in the definition of a Bessel system for $H$). Further, it is true of any two Banach spaces $X_1$, $X_2$ that the conditions (i) “$A$ is bounded below” and (ii) “$\ker A = \{0\}$ and $\text{ran} A$ is closed” are equivalent for a map $A \in bL(X_1, X_2)$. (We already used this in Proposition 5 of Lecture 3.) So, since $\ker T_X^*$ is trivial, and both $\ell_2(X)$ and $H$ are Banach spaces, (1) and (2) are equivalent.

1) $\implies$ 3) We only need to show that $T_XT_X^*$ is boundedly invertible since then, by the above discussion, $R = (T_XT_X^*)^{-1}$ does the job. In particular, $R$ being bounded, so is $T_{RX} = RT_X$, i.e., $RX$ is Bessel. For that, by assumption, $K := \inf_h \|T_X^*h\|/\|h\|$ is positive. Since

$$\|T_XT_X^*h\| \geq \frac{|\langle T_XT_X^*h, h \rangle|}{\|h\|} = \frac{\|T_X^*h\|^2}{\|h\|^2} \geq K^2\|h\| \quad \forall h \in H,$$

also the map $T_XT_X^*$ is bounded below, hence its kernel is trivial and its range is closed. But since it is self-adjoint, the closure of its range is the orthocomplement of its kernel, hence, all of $H$. In other words, $T_XT_X^*$ is boundedly invertible.

3) $\implies$ 1) Since $T_{RX} = (T_XT_{RX}^*)^* = \text{id}^* = \text{id}$, the map $T_X^*$ is bounded below (with $K = 1/\|T_{RX}\| > 0$ by (i)).
Definition 2. A Bessel system $X \subset H$ is called a frame if the conditions (1)-(3) of Theorem 1 above are satisfied.

Remark: The dual system $RX$ constructed in the above proof is the unique system in $H$ for which the projector $T_{RX}^*T_X = T_X^R T_X = T_X^T (T_X T_X^T)^{-1} T_X$, and its kernel is $\ker T_X$. However, unless $X$ is a Riesz basis, there are other dual frames in $H$ for $X$. Note that we could have introduced a super-space $H'$ in Proposition 5 of Lecture 3 without changing the conclusion of that proposition or its proof.

Exercise. Let $R$ be the map constructed in the proof of Theorem 1. Given $h \in H$, prove that $T_{RX}^* h$ has the least norm among all representations of $h$ as a combination of $X$.

Example 1. Let $X := \{x_1, \ldots, x_n\} \subset \mathbb{R}^m$, $m \leq n$. Then $T_X = [x_1, \ldots, x_n] \in \mathbb{R}^{m \times n}$ is a frame iff $T_X T^T = \mathbb{I}_{m \times m}$ for some $T \in \mathbb{R}^{n \times m}$ if the rows of the map $T_X$ are linearly independent iff $\text{rank} T_X = m$.

Theorem 3. Let $\phi \in L_2(\mathbb{R})$. The system $E(\phi)$ is a frame (for $S(\phi)$) iff

(i) $[\hat{\phi}, \hat{\phi}] \in L_\infty(\mathbb{R})$; and

(ii) $[\hat{\phi}, \hat{\phi}]^{-1} \in L_\infty(\Omega_\phi)$ with $\Omega_\phi := \text{supp} [\hat{\phi}, \hat{\phi}]$.

Proof: First note that

$$T_{E(\phi)}^* T_{E(\phi)} f \wedge = [\hat{f}, \hat{\phi}] \hat{\phi} \quad \forall \phi, \psi, f \in L_2(\mathbb{R}) \quad \text{s.t.} \quad [\hat{f}, \hat{\psi}] \hat{\phi} \in L_2(\mathbb{R}).$$

Indeed,

$$(T_{E(\phi)}^* T_{E(\phi)} f \wedge) = \left( \sum_{j \in \mathbb{Z}} \langle f, E^j \psi \rangle E^j \phi \right) = \sum_{j \in \mathbb{Z}} \langle f, E^j \psi \rangle e_{-j} \hat{\phi} = [\hat{f}, \hat{\psi}] \hat{\phi},$$

since $\sum_{j \in \mathbb{Z}} \langle f, E^j \psi \rangle e_{-j}$ is the Fourier series of $[\hat{f}, \hat{\psi}]$ (cf. Lecture 3).

$\Rightarrow$ If $E(\phi)$ is a frame, then, in particular, it is Bessel, so $[\hat{\phi}, \hat{\phi}] \in L_\infty(\mathbb{R})$ by Exercise 1 of Lecture 3. By Theorem 12 of Lecture 3, for any $f \in S(\phi)$, $\hat{f} = \tau \hat{\phi}$ for some $2\pi$-periodic $\tau$, hence, from (4),

$$\|T_{E(\phi)}^* f\|^2 = \langle T_{E(\phi)}^* T_{E(\phi)} f, f \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} [\hat{f}, \hat{\phi}] \hat{\phi} \overline{f} \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}, \hat{\phi}|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\tau \hat{\phi}|^2.$$

For $\varepsilon > 0$, let $\Omega_\varepsilon := \{ \xi \in \mathbb{T} : [\hat{\phi}, \hat{\phi}]^\wedge (\xi) \leq \varepsilon \}$. By Theorem 12 of Lecture 3, $f$ given implicitly by $\hat{f} = \chi_{\Omega_\varepsilon + 2\pi \mathbb{Z}} \hat{\phi}$ is in $S(\phi)$. Also, $\|f\|^2 = \frac{1}{2\pi} \|\hat{f}\|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} [\hat{\phi}, \hat{\phi}] \overline{f}$, while $\|T_{E(\phi)}^* f\|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}, \hat{\phi}|^2 = \frac{1}{2\pi} \int_{\Omega_\varepsilon} |[\hat{\phi}, \hat{\phi}]|^2$, hence $\|T_{E(\phi)}^* f\|^2 \leq \varepsilon \|f\|^2$. Since $T_{E(\phi)}^*$ is bounded below, this implies that $\text{meas} \Omega_\varepsilon = 0$ for some $\varepsilon > 0$.

$\Leftarrow$ Suppose the conditions (i) and (ii) hold. Define a function $\psi$ by

$$\hat{\psi}(\omega) = \begin{cases} \frac{\phi}{[\phi, \phi]}(\omega) & \text{if } \omega \in \Omega_\phi \\ 0 & \text{otherwise.} \end{cases}$$

This definition makes sense since, by (ii), the right hand side is in $L_2(\mathbb{R})$, so its inverse Fourier transform $\psi$ is defined and also lies in $L_2(\mathbb{R})$. Note also that, by Theorem 12 of Lecture 3, $\psi \in S(\phi)$. Let $f \in S(\phi)$. By that theorem, $\hat{f} = \tau \hat{\phi}$ for some $2\pi$-periodic $\tau$, hence (4) implies

$$(T_{E(\phi)} T_{E(\phi)}^*) \wedge = [\hat{f}, \hat{\phi}] \hat{\phi} = [\tau \hat{\phi}, \hat{\phi}] \hat{\phi} = [\frac{\tau}{[\phi, \phi]}, \hat{\phi}] [\phi, \phi] \hat{\phi} = [\tau \hat{\phi}, \hat{\phi}] = \tau \hat{\phi} = \hat{f}.$$

Thus, $T_{E(\phi)} T_{E(\phi)}^* = \text{id}$ on $S(\phi)$; by Theorem 1, $E(\phi)$ is a frame.

Example. Let $U$ be a measurable subset of $\mathbb{T}$ and let $\phi := \chi_U$. Then $[\hat{\phi}, \hat{\phi}] = \sum_{j \in \mathbb{Z}} \chi_U (\cdot - 2\pi j) \in L_\infty(\mathbb{R})$ and $[\hat{\phi}, \hat{\phi}]^{-1} \in L_\infty(U + 2\pi \mathbb{Z})$, so $E(\phi)$ is a frame. However, $E(\phi)$ is a Riesz basis only if $U = \mathbb{T}$, according to Proposition 8, part 2, of Lecture 3.