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Lecture 5: Principal shift-invariant (PSI) space theory

The space $S(\phi)$ generated by the shifts of $\phi \in L_2(\mathbb{R})$ is a closed subspace of $L_2(\mathbb{R})$, so there exists a unique orthogonal projector $P := P_\phi : L_2(\mathbb{R}) \to S(\phi)$ mapping each function $f \in L_2(\mathbb{R})$ to its ba (=: best approximation) $P_\phi f$ from $S(\phi)$. Recall that the element $P_\phi f$ is characterized by the conditions $P_\phi f \in S(\phi)$, $f - P_\phi f \perp S(\phi)$.

Theorem 1. *Let ϕ, $f \in L_2(\mathbb{R})$. Then $(P_\phi f)^\wedge = \frac{[\widehat{f},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}\widehat{\phi}$.*

Remarks:

(i) Here and hereafter we use the convention that zero times any extended number is zero, so if $[\widehat{\phi},\widehat{\phi}](\omega) = 0$, then $\widehat{\phi}(\omega) = 0$, hence $(P_\phi f)^\wedge(\omega) = 0$.

(ii) If $E(\phi)$ is a Riesz basis and $E(\psi)$ its dual basis with $\widehat{\psi} = \frac{\widehat{\phi}}{[\widehat{\phi},\widehat{\phi}]}$, then $T_{E(\phi)}T^*_{E(\psi)}$ is the orthogonal projector onto $S(\phi)$ and

$$(T_{E(\phi)}T^*_{E(\psi)}f)^\wedge = [\widehat{f},\widehat{\psi}]\widehat{\phi} = \frac{[\widehat{f},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}\widehat{\phi}.$$

Proof: Let $Q : f \mapsto (\frac{[\widehat{f},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}\widehat{\phi})^\vee$; show that $P_\phi = Q$. First note that Q is linear and bounded, the latter because

$$\|\frac{[\widehat{f},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}\widehat{\phi}\|^2_{L_2(\mathbb{R})} = \|\frac{|[\widehat{f},\widehat{\phi}]|^2}{[\widehat{\phi},\widehat{\phi}]^2}[\widehat{\phi},\widehat{\phi}]\|_{L_1(\mathbb{T})} \le \|\frac{[\widehat{f},\widehat{f}][\widehat{\phi},\widehat{\phi}]^2}{[\widehat{\phi},\widehat{\phi}]^2}\|_{L_1(\mathbb{T})} = \|[\widehat{f},\widehat{f}]\|_{L_1(\mathbb{T})} = \|\widehat{f}\|^2_{L_2(\mathbb{R})}$$

(with the inequality using the fact that, by the Cauchy-Buniakovsky-Schwarz inequality, $|[f,g]|^2 \le [f,f][g,g]$ $\forall f,g \in L_2(\mathbb{R})$), so $\|Q\| \le 1$.

Further, $f \perp S(\phi)$ iff $[\widehat{f},\widehat{\phi}]^\vee(j) = \langle f, E^j\phi\rangle = 0 \ \ \forall j \in \mathbb{Z}$ iff $[\widehat{f},\widehat{\phi}] = 0$ iff $Qf = 0$. Now,

$$(Q(E^j\phi))^\wedge = \frac{[e_{-\mathrm{i}j}\widehat{\phi},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}\widehat{\phi} = e_{-\mathrm{i}j}\widehat{\phi} = (E^j\phi)^\wedge,$$

so $Q = \mathrm{id}$ on the fundamental subset $E(\phi)$ of $S(\phi)$, hence $Q = \mathrm{id}$ on all $S(\phi)$. Thus, the action of P_ϕ and Q is the same on $S(\phi)$ as well as on its orthogonal complement. This means $P_\phi = Q$. □

As a corollary, we obtain the promised

Theorem 12 of Lecture 3. *Let $f, \phi \in L_2(\mathbb{R})$. Then $f \in S(\phi)$ iff $\widehat{f} = \tau\widehat{\phi}$ for some 2π-periodic function τ.*

Proof: "$\Longrightarrow$" If $f \in S(\phi)$, then $P_\phi f = f$, i.e., $\frac{[\widehat{f},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}\widehat{\phi} = \widehat{f}$. Both $[\widehat{f},\widehat{\phi}]$ and $[\widehat{\phi},\widehat{\phi}]$ are 2π-periodic, so one can define a 2π-periodic function τ so that $\tau = \frac{[\widehat{f},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}$ on $\mathrm{supp}[\widehat{\phi},\widehat{\phi}]$. Then $\tau\widehat{\phi} = \widehat{f}$.

"$\Longleftarrow$" If $\widehat{f} = \tau\widehat{\phi}$, then $(P_\phi f)^\wedge = \frac{[\tau\widehat{\phi},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}\widehat{\phi} = \tau\widehat{\phi} = \widehat{f}$, hence $P_\phi f = f$, hence $f \in S(\phi)$. □

Remark: The result we just proved looks quite natural in light of the observation that $(\sum_{j\in J} c(j)E^j\phi)^\wedge = \tau\widehat{\phi}$ with $\tau := \sum_{j\in J} c(j)e_{-\mathrm{i}j}$ a trigonometric polynomial if $J \subset \mathbb{Z}$ is finite.

As a further corollary, we find that, for $\operatorname{supp}\widehat{f} \subset \mathbb{T}$, $[\widehat{f}, \widehat{\phi}] = \widehat{f}\overline{\widehat{\phi}}$, hence

$$\|Pf\|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |\widehat{f}|^2|\widehat{\phi}|^2/[\widehat{\phi},\widehat{\phi}],$$

and therefore

$$\|f - Pf\|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |\widehat{f}|^2\Lambda_\phi^2, \quad \forall \operatorname{supp}\widehat{f} \subset \mathbb{T}, \tag{2}$$

with

$$\Lambda_\phi := (1 - |\widehat{\phi}|^2/[\widehat{\phi},\widehat{\phi}])^{1/2}.$$

The order of approximation. If one wants to determine how well a general function $f \in L_2(\mathbb{R})$ can be approximated by the elements of the scaled spaces $\sigma_h S(\phi) := \overline{\operatorname{span}(E^{jh}(\sigma_h\phi))}_{j\in\mathbb{Z}}$ (where $\sigma_h : f \mapsto f(\cdot/h)$), for a fixed function $\phi \in L_2(\mathbb{R})$, the natural thing to consider is the decay rate of $\operatorname{dist}_{L_2(\mathbb{R})}(f, \sigma_h S(\phi))$ when h tends to 0. In particular, one usually wants to know when the spaces $\sigma_h S(\phi)$ approximate suitably smooth functions to order h^k.

Definition 3. *Let $\phi \in L_2(\mathbb{R})$. The* **stationary PSI ladder associated with** *ϕ is the set $\mathcal{S}(\phi) := (\sigma_h S(\phi))_{h>0}$.*

Definition 4. *Given $\phi \in L_2(\mathbb{R})$, we say that $\mathcal{S}(\phi)$ (or simply $S(\phi)$) provides approximation order (a.o.) k if*

$$\operatorname{dist}_{L_2(\mathbb{R})}(f, \sigma_h S(\phi)) \le K h^k \|f\|_{W_2^k(\mathbb{R})} \quad \forall f \in W_2^k(\mathbb{R})$$

for some constant K depending only on ϕ.

Example 1. It was proven in the first part of the course that $S(\phi)$ provides a.o. k if ϕ is a cardinal B-spline of order k. □

For any $f = f_1 + f_2$,

$$\operatorname{dist}_2(f, \sigma_h S(\phi)) = \operatorname{dist}_2(f_1, \sigma_h S(\phi)) + O(\|f_2\|).$$

Consider the particular split $f = f_1 + f_2$, with

$$\widehat{f_1} = \chi_{\mathbb{T}/h}\widehat{f}.$$

Then

$$\|f_2\|^2 = \frac{1}{2\pi}\int_{\backslash\mathbb{T}/h} |\widehat{f}|^2 \le \frac{1}{2\pi}(\frac{h}{\pi})^{2k}\int_{\backslash\mathbb{T}/h}(1+|\cdot|)^{2k}|\widehat{f}|^2,$$

and the last integral goes to zero as $h \to 0$ in case $f \in W_2^{(k)}(\mathbb{R})$ since it is the tail end of $\int_{\mathbb{R}}(1+|\cdot|)^{2k}|\widehat{f}|^2 \sim \|f\|^2_{W_2^{(k)}(\mathbb{R})}$. Hence, for such f, $\|f_2\| = o(h^k)$, and therefore

$$\operatorname{dist}_2(f, \sigma_h S(\phi)) = \operatorname{dist}_2(f_1, \sigma_h S(\phi)) + o(h^k).$$

Next, observe that, for any $f \in L_2$ and $g \in S(\phi)$, $\|f - \sigma_h g\|^2 = h\|\sigma_{1/h}f - g\|$, hence $P_{\sigma_h S(\phi)} = \sigma_h P \sigma_{1/h}$. Therefore, for our choice of f_1,

$$\operatorname{dist}_2(f_1, \sigma_h S(\phi))^2 = \|f_1 - \sigma_h P\sigma_{1/h}f_1\|^2 = h\|(1-P)g\|^2,$$

with $g := \sigma_{1/h} f_1$, hence

$$\widehat{g} := (1/h)\sigma_h \chi_{\mathbb{T}/h}\widehat{f} = (1/h)\chi_{\mathbb{T}}\widehat{f}(\cdot/h),$$

and therefore, using (2),

$$\operatorname{dist}_2(f_1, \sigma_h S(\phi))^2 = \frac{1}{h2\pi}\int_{\mathbb{T}} |\widehat{f}(\cdot/h)|^2 \Lambda_\phi^2.$$

If now $\Lambda_\phi/|\cdot|^k$ is essentially bounded on $\mathbb{T}$, then

$$\operatorname{dist}_2(f_1, \sigma_h S(\phi))^2 \le \frac{\|\Lambda_\phi/|\cdot|^2\|_{L_\infty(\mathbb{T})}}{2\pi}\int_{\mathbb{T}} |\cdot|^{2k}|\widehat{f}(\cdot/h)|^2 \,\mathrm{d}(\cdot/h) \le O(h^{2k}\|f\|_{W_2^{(k)}(\mathbb{R})}).$$

If, on the other hand, $\Lambda_\phi/|\cdot|^k$ fails to be essentially bounded on $\mathbb{T}$, then it is possible to show that

$$\frac{1}{h2\pi}\int_{\mathbb{T}} |\widehat{f}(\cdot/h)|^2 \Lambda_\phi^2$$

cannot be $O(h^{2k})$ for every $f \in W_2^{(k)}$ (for details, see [C. de Boor, R. DeVore, A. Ron *Approximation from shift-invariant subspaces of* $L_2(\mathbb{R}^d)$] available as the `postscript` file `l2shift.ps` from the site `/ftp.cs.wisc.edu` in the subdirectory `Approx`). This gives

Theorem 5. *Let* $\phi \in L_2(\mathbb{R})$. *The space* $S(\phi)$ *provides a.o.* k *iff* $\Lambda_\phi/|\cdot|^k$ *is essentially bounded on* $\mathbb{T}$.

Since $0 \le \Lambda_\phi \le 1$, this essential boundedness is an issue only in a neighborhood of 0.

Now notice that $\Lambda_\phi^2 = M/(|\widehat{\phi}|^2 + M)$, with $M := \sum_{\alpha\in 2\pi\mathbb{Z}\backslash 0} |E^\alpha \widehat{\phi}|^2$. Hence, if $0 < K_2 \le |\widehat{\phi}| \le K_1 < \infty$ in some neigborhood B of zero (this condition holds, e.g., if $\widehat{\phi}(0) \ne 0$ and $\phi \in L_1(\mathbb{R})$, so $\widehat{\phi}$ is continuous), then $\Lambda_\phi = O(|\cdot|^k)$ if and only if $M = O(|\cdot|^{2k})$.

Corollary 6. *Let* $\phi \in L_2(\mathbb{R})$ *be compactly supported and* $\widehat{\phi}(0) \ne 0$. *Then* $\mathcal{S}(\phi)$ *provides a.o.* k *iff* $\widehat{\phi}$ *has a* k*-fold zero at each* $\alpha \in 2\pi\mathbb{Z}\setminus\{0\}$.

Proof: "$\Longrightarrow$" If $\mathcal{S}(\phi)$ provides a.o. k, then, by the argument we just gave, M has a $2k$-fold zero at the origin, hence $E^\alpha(|\widehat{\phi}|^2) \le M = O(|\cdot|^{2k})\ \ \forall \alpha \in 2\pi\mathbb{Z}\setminus\{0\}$.

"$\Longleftarrow$" Let $B := [-\delta \,..\, \delta]$ be a neighborhood of zero where $\widehat{\phi}$ is bounded away from zero. Since ϕ is compactly supported, $\widehat{\phi} \in W_2^{(l)}(\mathbb{R})\ \ \forall l \in \mathbb{N}$. Let $l := k+1$. By the Taylor formula, $|\widehat{\phi}(\omega-\alpha)| \le \|D^k\widehat{\phi}\|_{L_\infty(B-\alpha)}|\omega|^k/k!$, hence $E^\alpha(|\widehat{\phi}|^2)(\omega) \le \left(\frac{1}{k!}\right)^2\|D^k\widehat{\phi}\|^2_{L_\infty(B-\alpha)}|\omega|^{2k}$ for all $\alpha \in 2\pi\mathbb{Z}\setminus\{0\}$ and any $\omega \in B$. Since $\|D^k\widehat{\phi}\|_{L_\infty(B-\alpha)} \le C\|\widehat{\phi}\|_{W_2^{(k+1)}(B-\alpha)}$ with a constant C depending only on k and δ, we have

$$M(\omega) \le C'|\omega|^{2k}\sum_{\alpha\in 2\pi\mathbb{Z}\setminus\{0\}} \|\widehat{\phi}\|^2_{W_2^{(k+1)}(B-\alpha)} \le C'|\omega|^{2k}\|\widehat{\phi}\|^2_{W_2^{(k+1)}(\mathbb{R})}$$

for some constant C' depending only on k and δ, i.e., $M = O(|\cdot|^{2k})$, hence $\Lambda_\phi = O(|\cdot|^k)$. □

Recall that the Strang-Fix conditions of order k, $D^\gamma\widehat{\phi}(\alpha) = 0\ \ \forall\alpha \in 2\pi\mathbb{Z}\setminus\{0\}\ \ \forall\gamma = 0,\ldots,k-1$ and $\widehat{\phi}(0) \ne 0$, are equivalent to the reproduction of the polynomials of degree $< k$ under the map $\phi*' : f \mapsto \sum_{\alpha\in\mathbb{Z}} f(\alpha)E^\alpha\phi$ (see Theorem 2.5 of Lecture 2). Therefore,

Corollary 7. *Let* $\phi \in L_2(\mathbb{R})$ *be compactly supported and let* $\widehat{\phi}(0) \ne 0$. *Then* $\mathcal{S}(\phi)$ *provides a.o.* k *iff* $\phi*'$ *reproduces* $\Pi_{<k}$.

Example 2. If $\widehat{\phi} = \chi_{\mathbb{T}}$ (i.e., $\phi(x) = \frac{\sin(\pi x)}{\pi x}$), then, for this ϕ, $\Lambda_\phi = 0$ around the origin, and hence $S(\phi)$ provided all possible approximation orders. This is related to the fact that ϕ satisfies the SF conditions of all orders. At the same time, one cannot talk about polynomial reproduction here, since ϕ does not decay at ∞ fast enough. This shows that the notion of the SF conditions is more fundamental that the (sometimes equivalent) notion of polynomial reproduction. That might come as a surprise since the original argument for approximation orders (as presented by Carl de Boor, in his lectures) uses polynomial reproduction as the vehicle, with the SF conditions added as an incidental observation.