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Lecture 6: Linear independence of $E(\phi)$

Let $\mathcal{Q}$ be the space $\mathbb{C}^{\mathbb{Z}}$ with the topology of pointwise convergence, i.e., with the topology given by the seminorms $\|q\|^{(n)} := \max_{|j|\le n} |q(j)|$, for $n \in \mathbb{N}$. Given a compactly supported function ϕ, let $S_*(\phi)$ denote the range of $\phi*'$ as a map on $\mathcal{Q}$, i.e., the space of *all* linear combinations $\sum_{j\in\mathbb{Z}} c(j)E^j\phi$ with complex coefficients. There is no problem of pointwise convergence of such sums since, for any point $x \in \mathbb{R}$, only finitely many terms of the sum are nonzero at x. We say that the shift sequence $E(\phi)$ is **linearly independent (LI)** if $\phi*'$ is injective (on $\mathcal{Q}$).

Exercise 1. For a compactly supported ϕ, the map $(\phi*')|_{l_2(\mathbb{Z})}$ is always injective.

Exercise 2. The space $\mathcal{Q}$ is Fréchet, the pairing

$$\ell_0 \times \mathcal{Q} \ni (p,q) \mapsto \sum_{j\in\mathbb{Z}} p(j)q(j)$$

makes $\ell_0 := \ell_0(\mathbb{Z})$ into the continuous dual of $\mathcal{Q}$ and $\mathcal{Q}$ an algebraic dual of ℓ_0. It follows from the general Hahn-Banach theorem that $(K^\perp)_\perp = K$ (where, for sure, $K^\perp := \{p \in \ell_0 : (p,q) = 0 \ \ \forall q \in K\}$, $(K^\perp)_\perp := \{q \in \mathcal{Q} : (p,q) = 0 \ \ \forall p \in K^\perp\}$) for any closed linear subspace K of $\mathcal{Q}$.

Theorem 1. *Let ϕ be a compactly supported distribution (hence $\widehat{\phi}$ is entire). Then $E(\phi)$ is LI iff $\widehat{\phi}$ does not have a 2π-periodic zero in $\mathbb{C}$.*

Proof: "$\Longrightarrow$" By Theorem 2.6 of Lecture 2, $\widehat{\phi}|_{\theta+2\pi\mathbb{Z}} = 0$ iff $\phi *' e_{i\theta} = 0$, so if $\widehat{\phi}$ has a 2π-periodic zero in $\mathbb{C}$, then the map $\phi*'$ is not injective.

"$\Longleftarrow$" Suppose $K := \ker(\phi*') \ne \{0\}$. Then K is a closed linear subspace of $\mathcal{Q}$, therefore $K = (K^\perp)_\perp$. Since $\phi *' (Ec) = E(\phi *' c)$, K is shift-invariant, hence $K^\perp$ is a shift-invariant subspace of ℓ_0. Let

$$K_+^\perp := \{p^\vee := \sum_{j\in\mathbb{Z}} p(j)()^j : p \in K^\perp, \ \ p(j) = 0 \ \ \forall j < 0\}.$$

$K_+^\perp$ is a linear subspace of Π. Further, since $()^1 p^\vee = (Ep)^\vee$, $K_+^\perp$ is closed under multiplication by polynomials, hence an ideal in Π. Since the ring Π is a principal ideal domain, we have $K_+^\perp = p_0^\vee \Pi$ for some $p_0^\vee \in \Pi$. Therefore,

$$K^\perp = \operatorname{span}(E(p_0)).$$

In particular, if $p_0^\vee$ were a monomial, then $p_0 = \delta_{j_0}$ for some $j_0 \in \mathbb{Z}$, so $K^\perp = \ell_0$, hence $K = (K^\perp)_\perp = \{0\}$. Contradiction! So, $p_0^\vee$ is not a monomial and, therefore, has a nonzero root, ξ say. Then $0 = p_0^\vee(\xi) = \sum_{j\in\mathbb{Z}} p_0(j)\xi^j =: (p_0, \xi^{\mathbb{Z}})$, therefore

$$0 = \xi^k (p_0, \xi^{\mathbb{Z}}) = (E^k p_0, \xi^{\mathbb{Z}}) \quad \forall k \in \mathbb{Z}.$$

But this says that $\xi^{\mathbb{Z}} \perp K^\perp$, hence $\xi^{\mathbb{Z}} \in K$ and, choosing $\theta \in \mathbb{C}$ so that $\xi = e^{i\theta}$ (since $\xi \ne 0$, can always solve this equation), we are done, by Theorem 2.6 of Lecture 2. □

The above theorem shows that, for a compactly supported function $\phi \in L_2(\mathbb{R})$, the condition "$E(\phi)$ is LI" is stronger than "$E(\phi)$ is a Riesz basis of $S(\phi)$".

Example 2. If $\phi := \chi_{[0..1]}$, then the set of zeros of $\widehat{\phi} : \omega \mapsto \frac{1-\mathrm{e}^{-\mathrm{i}\omega}}{\mathrm{i}\omega}$ is $2\pi\mathbb{Z} \setminus \{0\}$, so $E(\phi)$ is LI (and is a Riesz basis). For $\psi := \chi_{[0..2]}$, its Fourier transform $\widehat{\psi} : \omega \mapsto \frac{1-\mathrm{e}^{-2\mathrm{i}\omega}}{\mathrm{i}\omega}$ has a 2π-periodic real zero $\pi + 2\pi\mathbb{Z}$, so $E(\psi)$ is not a Riesz basis (and not LI). Finally, if $\eta := \chi_{[0..1]} + \beta\chi_{[1..2]}$ for some $|\beta| < 1$, then $\widehat{\eta} : \omega \mapsto \left(1 + \beta\mathrm{e}^{-\mathrm{i}\omega}\right) \frac{1-\mathrm{e}^{-\mathrm{i}\omega}}{\mathrm{i}\omega}$ has a 2π-periodic complex zero but no 2π periodic real ones, so $E(\eta)$ is a Riesz basis but not LI.

Factorization of generators for local PSI spaces. Local PSI spaces were defined in Lecture 1 as the spaces where compactly supported functions are dense. Here we shall use an equivalent definition: a PSI space is local if it has a compactly supported generator. The following lemma shows that in such spaces one can always replace a "bad" generator (whose shift sequence is not LI) by a "good" one (whose shift sequence is LI).

Lemma 3. *Let $\phi \neq 0$ be a distribution with $\operatorname{supp}\phi \subseteq [a\,..\,b]$, a and b finite. Then there exists a distribution $\psi \in S_*(\phi)$ s.t.*
(i) ϕ is finitely spanned by $E(\psi)$, i.e., ϕ is a finite linear combination of the shifts of ψ.
(ii) $E(\psi)$ is LI.
(iii) $\operatorname{supp}\psi \subseteq [a\,..\,b - \dim\ker(\phi*')]$.

Remark: (iii) says, in particular, that $\dim\ker(\phi*') < \infty$.
Proof: Assume that $E(\phi)$ is not LI (otherwise $\psi := \phi$ possesses all the desired properties). Then, by Theorem 2.6 of Lecture 2, $\phi *' e_{\mathrm{i}\theta_1} = 0$ for some $\theta_1 \in \mathbb{C}$. Define

$$\phi_1 := \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\theta_1} E^j\phi \qquad \left(= -\sum_{j=-\infty}^{-1} \mathrm{e}^{\mathrm{i}j\theta_1} E^j\phi\right) \tag{4}$$

and observe that $E^1\phi_1 = \sum_{j=0}^{\infty} \mathrm{e}^{\mathrm{i}j\theta_1} E^{j+1}\phi = \mathrm{e}^{-\mathrm{i}\theta_1}\left(\sum_{j=1}^{\infty} \mathrm{e}^{\mathrm{i}j\theta_1} E^j\phi\right)$, hence $\phi_1 - \mathrm{e}^{\mathrm{i}\theta_1}\phi_1 = \phi$, so ϕ is a combination of two shifts of ϕ_1. Since $\operatorname{supp} E^j\phi \subseteq [a+j\,..\,b+j]$, the first equality of (4) implies $\operatorname{supp}\phi_1 \subseteq [a\,..\,+\infty)$ and the second equality that $\operatorname{supp}\phi_1 \subseteq (-\infty\,..\,b-1]$. So, $\operatorname{supp}\phi_1 \subseteq [a\,..\,b-1]$. Repeat this procedure to get $\phi_2, \ldots, \phi_n =: \psi$ with $E(\phi_n)$ LI (such an n always exists and does not exceed $b-a$, since $\operatorname{supp}\phi_j \subseteq [a\,..\,b-j]$ and since the condition $\operatorname{supp}\nu \subseteq [a\,..\,a+1)$ implies the linear independence of $E(\nu)$; – see Lemma 2.1 of Lecture 2). From the construction, $\phi = \sum_{j=0}^{n} a(j)E^j\psi$ for some $a \in \mathbb{C}^{n+1}$ with $a(0) \neq 0 \neq a(n)$. For any $c \in \mathbb{C}^{\mathbb{Z}}$, we have

$$\phi *' c = \sum_{k\in\mathbb{Z}} c(k) \sum_{j=0}^{n} a(j)E^{k+j}\psi = \sum_{k\in\mathbb{Z}} \left(\sum_{j=0}^{n} c(k-j)a(j)\right) E^k\psi,$$

so if $\phi *' c = 0$, then, by the linear independence of $E(\psi)$,

$$\sum_{j=0}^{n} c(k-j)a(j) = 0 \qquad \forall k \in \mathbb{Z}. \tag{5}$$

This system, for $c \in \mathcal{Q}$, has exactly n linearly independent solutions, since, after $c(0), \ldots, c(n-1)$ are fixed, the equation (5) applied to $k = n$ uniquely determines $c(n)$, then (5) with $k = n+1$ uniquely determines $c(n+1)$ and so on; similarly, (5) applied to $k = n-1, n-2, \ldots$ uniquely determines $c(-1)$, $c(-2)$ etc. So, $\dim\ker(\phi*') = n$, which completes the proof of (iii). □

This lemma says $\phi = \psi *' c$ with $\# \operatorname{supp} c < \infty$ and $E(\psi)$ LI, so $\widehat{\phi} = \widehat{c}\widehat{\psi}$ and, by Theorem 1, $\widehat{\psi}$ has no 2π-periodic zeros in $\mathbb{C}$. In other words, after dividing $\widehat{\phi}$ by a suitable trigonometric polynomial $\widehat{c}$ whose set of 2π-periodic zeros is the same as that of $\widehat{\phi}$, one obtains a quotient $\widehat{\psi}$ without any 2π-periodic zeros, and so $E(\psi)$ is LI.

Here is another kind of factorization of the generator ϕ.

Lemma 6. *Suppose ϕ is a function with $\operatorname{supp} \phi \subseteq [a \mathinner{..} b]$, a and b finite. If $\Pi_{k-1} \subseteq S_*(\phi)$, then there exists a distribution ψ with $\operatorname{supp} \psi \subseteq [a \mathinner{..} b-k]$ s.t. $\phi = B_k * \psi$.*

Proof: First show that $\phi *' \Pi_{k-1} \subseteq \Pi_{k-1}$. Indeed, let $p \in \Pi_{k-1}$. Then, by the hypothesis of the lemma, $p = \phi *' c_p$ for some $c_p \in \mathbb{C}^{\mathbb{Z}}$. However, since, for any $j \in \mathbb{Z}$, $(\phi *' c)(j) = (\phi|_{\mathbb{Z}} * c)(j) = (c * \phi|_{\mathbb{Z}})(j) = (c *' \phi)(j)$, we have

$$\phi *' (\phi *' c_p) = \phi *' (c_p *' \phi) = (\phi *' c_p) *' \phi = p *' \phi \ = \ \sum_j E^j p\, \phi(j) \in \Pi_{k-1}$$

since ϕ has compact support and each $E^j p$ belongs to Π_{k-1}. Now, let ϕ_0 be the kth distributional derivative of ϕ. Then $\operatorname{supp} \phi_0 \subseteq \operatorname{supp} \phi$ and

$$\phi_0 * \Pi_{k-1} = (D^k \phi) * \Pi_{k-1} = D^k(\phi * \Pi_{k-1}) \subseteq D^k \Pi_{k-1} = \{0\}.$$

It follows from the proofs of Theorems 2.5 and 2.6 of Lecture 2 that $\widehat{\phi_0}$ has a k-fold zero at each $\alpha \in 2\pi\mathbb{Z}$, so we can apply the procedure from the proof of Lemma 3 to $\theta_1 = \cdots = \theta_k = 0$. This gives a sequence of distributions $\phi_1, \ldots, \phi_k$ s.t. $\Delta\phi_l = \phi_{l-1}$, $\operatorname{supp} \phi_l \subseteq [a \mathinner{..} b-l]$, $l = 1, \ldots, k$. So, $\Delta^k \phi_k = \phi_0 = D^k \phi$. But $\Delta^k \phi_k = B_0 * \delta^k \phi_k = D^k(B_k * \phi_k)$ by the formula (1.3) of Lecture 1 (with B_0 the Dirac δ-function), so $B_k * \phi_k - \phi \in \Pi_{k-1}$. Since both ϕ and ϕ_k have compact support, we get $B_k * \phi_k = \phi$. □

Combining Lemma 3 and Lemma 6, we get

Factorization Theorem 7. *Let ϕ be a function with $\operatorname{supp} \phi \subseteq [a \mathinner{..} b]$, a and b finite. Then there exists $k \in \mathbb{Z}_+$, $c \in \ell_0$, and a distribution $\psi \in S_*(\phi)$ with $\operatorname{supp} \psi \subseteq [a \mathinner{..} b - k - \dim \ker(\phi *')]$ s.t.*
(i) $\phi = (\psi * B_k) *' c$,
(ii) $\Pi \cap S_*(\phi) = \{0\}$,
(iii) $E(\psi)$ *is LI.*

Proof: By the proof of Lemma 3, there exists a function (not just a distribution) $\eta \in S_*(\phi)$ s.t. ϕ is finitely spanned by $E(\eta)$, $E(\eta)$ is LI, and $\operatorname{supp} \eta \subseteq [a \mathinner{..} b - \dim \ker(\phi *')]$. Let $k := \sup\{l : \eta = B_l * \xi$ for some distribution ξ with $\operatorname{supp} \subseteq [a \mathinner{..} b - \dim \ker(\phi *') - l]\}$ and let $\psi \in S_*(\eta) (= S_*(\phi))$ be s.t. $\eta = B_k * \psi$. If $p \in \Pi \cap S_*(\psi)$, then, by the shift-invariance of $S_*(\phi)$, $\Delta^l p \in \Pi \cap S_*(\phi) \quad \forall l = 0, \ldots, \deg p$; since $\deg \Delta^l p = \deg p - l$, the sequence $(\Delta^l p)_{l=0}^{\deg p}$ is LI, hence $\Pi_{\le \deg p} \subseteq S_*(\phi)$. If $\Pi \cap S_*(\psi) \neq \{0\}$, then, by Lemma 6, $\psi = B_1 * \psi_1$ for some ψ_1 with $\operatorname{supp} \psi_1 \subseteq [a \mathinner{..} b - \dim \ker(\phi *') - k - 1]$, contrary to the maximality of k. So, $\Pi \cap S_*(\psi) = \{0\}$. Since $\ker(\psi *') \subseteq \ker(\eta *')$ and $E(\eta)$ is LI, so is $E(\psi)$. □

One naturally would like that a generator of our PSI space be a distribution with minimal support, i.e., $\operatorname{diam} \operatorname{supp} \phi \le \operatorname{diam} \operatorname{supp} f$ for any $f \in S_*(\phi)$. Other nice properties of ϕ that we want are the linear independence of its shift sequence and finite spanning of each compactly supported distribution in $S_*(\phi)$. Finally, if the generator ψ of the system $E(\psi)$ dual to $E(\phi)$ were a compactly supported infinitely smooth function, it would facilitate working with the analysis operator. As it turns out, all these properties are equivalent for a compactly generator of a PSI space. Precisely,

Theorem 8. *Given a compactly supported distribution ϕ, TFAE:*
(i) $E(\phi)$ *is LI.*
(ii) There exists $\psi \in \mathcal{D}(\mathbb{R})$ *s.t.* $\langle \psi, E^j \phi \rangle = \delta_{j0}$.
(iii) Each compactly supported distribution $f \in S_*(\phi)$ *is finitely spanned by* $E(\phi)$.
(iv) ϕ *has minimal support.*
(v) $\widehat{\phi}$ *does not have a* 2π*-periodic zero in* $\mathbb{C}$.

Remark: As usual, $\mathcal{D}(\mathbb{R})$ denotes the Fréchet space of test function and $\mathcal{D}'(\mathbb{R})$ that of distributions.

Proof: We proved the equivalence of (i) and (v) in Theorem 1.

(i)$\Longrightarrow$(ii) The map $\phi *' : \mathcal{Q} \to \mathcal{D}'(\mathbb{R})$ is continuous due to the compact support of ϕ. The dual of $\mathcal{D}'$ is $\mathcal{D}$ (see, e.g., [1, Theorem XIV, p.75]); the dual of $\mathcal{Q}$ can be identified (via the pairing introduced in Exercise 2) with ℓ_0 equipped with the following topology: a sequence $(p_k) \in \ell_0^{\mathbb{N}}$ converges to p if $\#(J := \cup_{k \in \mathbb{N}} \operatorname{supp} p_k) < \infty$ and $p_k \to p$ pointwise on J. It follows that $(\phi *')^* : \mathcal{D}(\mathbb{R}) \to \ell_0 : f \mapsto (\langle f, E^j \phi \rangle)_{j \in \mathbb{Z}}$. Since $\operatorname{ran}((\phi *')^*)^- = \ker(\phi *')^\perp = \{0\}^\perp = \ell_0$ (check!), there exists $\psi \in \mathcal{D}(\mathbb{R})$ s.t. $(\langle \psi, E^j \phi \rangle)_{j \in \mathbb{Z}} = (\phi *')^* \psi = \delta_0$, and (ii) follows.

(ii)$\Longrightarrow$(iii) If $f \in S_*(\phi)$ has compact support, then at most finitely many $\langle f, E^j \psi \rangle$ are nonzero, so $f = \sum_{j \in \mathbb{Z}} \langle f, E^j \psi \rangle E^j \phi$ is a finite linear combination of $E(\phi)$.

(iii)$\Longrightarrow$(iv) If $f \in S_*(\phi) \neq 0$ is compactly supported, then $f = \phi *' c$ with $\operatorname{supp} c \subseteq (\alpha, \ldots, \beta)$ where α, $\ldots$, β are consecutive integers, $c(\alpha) \neq 0$, and $c(\beta) \neq 0$. If $[a \mathinner{..} b]$ is the minimal closed interval containing $\operatorname{supp} \phi$, then $[a+\alpha \mathinner{..} b+\beta]$ is the minimal closed interval containing $\operatorname{supp} f$, so $\operatorname{diam} \operatorname{supp} f = b - a + \alpha - \beta \geq b - a = \operatorname{diam} \phi$.

(iv)$\Longrightarrow$(i) By Lemma 3, there exists a compactly supported distribution $f \in S_*(\phi)$ with $E(f)$ LI s.t. $\phi = f *' c$ where $\# \operatorname{supp} c < \infty$. As we just saw, this implies $\operatorname{diam} \operatorname{supp} \phi = \operatorname{diam} \operatorname{supp} f + \operatorname{diam} \operatorname{supp} c - 1$, so $\operatorname{diam} \operatorname{supp} c = 1$, i.e., c is a multiple of a delta-sequence. Therefore, $E(\phi)$ was originally LI. □

Remark: The above proof also shows that the generator with the properties listed in the theorem is unique up to multiplication by a constant and shifts.

Theorem 9. *Let ϕ be a compactly supported distribution. TFAE:*

(i) *$\widehat{\phi}$ does not have a 2π-periodic zero in $\mathbb{R}$.*

(ii) *$\ker(\phi *')$ contains no sequences of slow growth (a sequence c has* **slow growth** *if $|c(j)| \leq |p(j)|$ for some $p \in \Pi$).*

(iii) *$\phi *'$ is 1-1 on $l_\infty(\mathbb{Z})$.*

Proof: (i)$\Longrightarrow$(ii) Let $c \in \ker(\phi *')$ be a sequence of slow growth. Then $\tilde{c} := \sum_{j \in \mathbb{Z}} c(j) \delta_j$ is the second derivative of the continuous function $\sum_{j \in \mathbb{Z}} c(j)(\cdot - j)_+$ of slow growth, hence a tempered distribution (see [1, Theorem VI, p.239]). By the Paley-Wiener Theorem (see [2, Theorem 7.23, p.183]), $\widehat{\phi}$ is an entire function of slow growth, hence $\widehat{\tilde{c}} \widehat{\phi}$ is a tempered distribution equal to $(\phi *' c)^\wedge = 0$, hence $\operatorname{supp} \widehat{\tilde{c}}$ is contained in the set $\mathcal{Z}(\widehat{\phi})$ of zeros of $\widehat{\phi}$. The distribution $\widehat{\tilde{c}}$ is 2π-periodic and so is its support. Since $\widehat{\phi}$ has no 2π-periodic real zeros, this implies $\operatorname{supp} \widehat{\tilde{c}} = 0$, hence $\tilde{c} = 0$, hence $c = 0$.

(ii)$\Longrightarrow$(iii) Trivial.

(iii)$\Longrightarrow$(i) If $\widehat{\phi}|_{\theta + 2\pi \mathbb{Z}} = 0$ for $\theta \in \mathbb{R}$, then (Theorem 2.6 of Lecture 2) $\phi *' e_{\mathrm{i}\theta} = 0$. Since $e_{\mathrm{i}\theta}|_{\mathbb{Z}} \in l_\infty(\mathbb{Z})$, this finishes the proof. □

Final comments. 1. If ϕ is a compactly supported $L_2(\mathbb{R})$-function, then we could add two more equivalent conditions to the previous theorem: (iv) $E(\phi)$ is a Riesz basis, (v) $E(\phi)$ has a dual basis $E(\psi)$ with ψ a function exponentially decaying at $\pm\infty$.

2. Here is a brief summary on the bracket product. With ϕ and $\psi \in L_2(\mathbb{R})$, $[\widehat{\phi}, \widehat{\phi}] = 1$ iff $E(\phi)$ are orthonormal; $[\widehat{\phi}, \widehat{\psi}] = 1$ iff $E(\phi)$ and $E(\psi)$ are biorthonormal; $[\widehat{\phi}, \widehat{\psi}] = 0$ iff $S(\phi) \perp S(\psi)$.

3. A similar, though more sophisticated, theory exists for finitely generated shift-invariant (FSI) spaces with more than one generator. Time does not permit us to go into pertinent details. If, later on, we need results from FSI space theory, they will be stated and the reader referred to the corresponding literature.

References

[1] Laurent Schwartz *Théory des distributions*. – Paris, Hermann, 1966.

[2] Walter Rudin *Functional Analysis*. - New York, McGraw-Hill, 1973.