Generalized B-splines and local refinements

Carla Manni

Department of Mathematics, University of Roma “Tor Vergata”

collaboration with

P. Costantini, F. Pelosi, H. Speleers

11-th MAIA Conference

September 25–30, 2013

“Ettore Majorana” Foundation and Centre, Erice
Outline

- Bernstein-like representations
Outline

- Bernstein-like representations
- Generalized B-splines
Outline

- Bernstein-like representations
- Generalized B-splines
- Local refinements
Outline

- Bernstein-like representations
- Generalized B-splines
- Local refinements
  - Hierarchical bases for Generalized B-splines
Outline

- Bernstein-like representations
- Generalized B-splines
- Local refinements
  - Hierarchical bases for Generalized B-splines
  - Generalized B-splines over T-meshes
Outline

- Bernstein-like representations
- Generalized B-splines
- Local refinements
  - Hierarchical bases for Generalized B-splines
  - Generalized B-splines over T-meshes
  - Generalized B-splines over triangles
Outline

- Bernstein-like representations
- Generalized B-splines
- Local refinements
  - Hierarchical bases for Generalized B-splines
  - Generalized B-splines over T-meshes
  - Generalized B-splines over triangles
- Ariadne’s thread
CAGD: Bézier forms

\[ \sum_{i=0}^{p} p_i \binom{p}{i} t^i (1 - t)^{p-i}, \quad t \in [0, 1], \; p_i \in \mathbb{R}^d \]
CAGD: Bézier forms

\[\sum_{i=0}^{p} p_i \binom{p}{i} t^i (1 - t)^{p-i}, \quad t \in [0, 1], \quad p_i \in \mathbb{R}^d\]

BÉZIER CURVE
CAGD: Bézier forms

\[ \sum_{i=0}^{p} p_i \binom{p}{i} t^i (1 - t)^{p-i}, \quad t \in [0, 1], \; p_i \in \mathbb{R}^d \]

**BÉZIER CURVE**
CAGD: Bézier forms

\[ \sum_{i=0}^{p} p_i \binom{p}{i} t^i (1 - t)^{p-i}, \quad t \in [0, 1], \ p_i \in \mathbb{R}^d \]

BÉZIER CURVE

space: \( \mathbb{P}_p \)
basis: Bernstein pol.
CAGD: Bernstein like representation

$$\sum_{i=0}^{p} p_i B_i(t), \quad t \in [a, b], \quad p_i \in \mathbb{R}^d$$
CAGD: Bernstein like representation

\[ \sum_{i=0}^{p} p_i B_i(t), \quad t \in [a, b] \quad p_i \in \mathbb{R}^d \]
CAGD: Bernstein like representation

\[ \sum_{i=0}^{p} p_i B_i(t), \quad t \in [a, b] \quad p_i \in \mathbb{R}^d \]

Bernstein-like representation

space: \(<B_0, \cdots, B_n>\)
CAGD: Bernstein like representation

\[ \sum_{i=0}^{p} p_i B_i, \quad t \in [a, b] \quad p_i \in \mathbb{R}^d \]

space: \( < B_0, \cdots, B_n > \)

basis: ONTP basis (B-basis)

Optimal Normalized Totally Positive
Bernstein/B-splines \Rightarrow \text{Optimal NTP bases}

- Bernstein/B-splines bases are the ONTP bases for polynomials/piecewise polynomials

\[ \downarrow \]

- optimal from a geometric point of view
- optimal from a computational point of view
Bernstein/B-splines ⇒ Optimal NTP bases

- Bernstein/B-splines bases are the ONTP bases for polynomials/piecewise polynomials

  ↓

- optimal from a geometric point of view
- optimal from a computational point of view
Beyond polynomials: constrained curves/surfaces

in CAGD curves/surfaces are often subjected to constraints
Beyond polynomials: constrained curves/surfaces

- in CAGD curves/surfaces are often subjected to constraints
  - reproduction constraints
    exact reproduction of main curves/surfaces (conic sections, ...)
  - shape constraints
    curvature orientation, torsion signs,...
  - tolerance constraints
    offset constraints,...
  - ...

Generalized B-splines and local refinements – p. 6/50
Beyond polynomials: constrained curves/surfaces

in CAGD curves/surfaces are often subjected to constraints

**reproduction constraints**
exact reproduction of main curves/surfaces (conic sections, ...)

**shape constraints**
curvature orientation, torsion signs,...

**tolerance constraints**
offset constraints,...

...
Reproducing conic sections, cycloids ....

- exponentials $< 1, t, e^{\omega t}, e^{-\omega t} >$
Reproducing conic sections, cycloids ....

- **exponentials** \(< 1, t, e^{\omega t}, e^{-\omega t} >\)
  - \(\omega\) : shape parameter
  - cubic as \(\omega \to 0\)
  - linear as \(\omega \to +\infty\)
Reproducing conic sections, cycloids ....

- **exponentials** $< 1, t, e^{\omega t}, e^{-\omega t} >$
  - $\omega$: shape parameter
  - cubic as $\omega \to 0$
  - linear as $\omega \to +\infty$

- **trigonometrics** $< 1, t, \cos(\omega t), \sin(\omega t) >$
Reproducing conic sections, cycloids ....

- **exponentials** $< 1, t, e^{\omega t}, e^{-\omega t} >$
  - $\omega$ : shape parameter
  - cubic as $\omega \to 0$
  - linear as $\omega \to +\infty$

- **trigonometrics** $< 1, t, \cos(\omega t), \sin(\omega t) >$
  - $\omega$ : shape parameter
  - cubic if $\omega \to 0$
Shape constraints

- exponentials $< 1, t, e^{\omega t}, e^{-\omega t}>$
Shape constraints

- **exponentials** \( < 1, t, e^{\omega t}, e^{-\omega t} > \)
  - \( \omega \) : shape parameter
  - cubic as \( \omega \to 0 \)
  - linear as \( \omega \to +\infty \)
Shape constraints

- **exponentials** \( < 1, t, e^{\omega t}, e^{-\omega t} > \)
  - \( \omega \) : shape parameter
  - cubic as \( \omega \to 0 \)
  - linear as \( \omega \to +\infty \)

- **variable degree** \( < 1, t, t^\omega, (1 - t)^\omega > \)
Shape constraints

exponentials $< 1, t, e^{\omega t}, e^{-\omega t} >$

- $\omega$: shape parameter
- cubic as $\omega \to 0$
- linear as $\omega \to +\infty$

variable degree $< 1, t, t^\omega, (1 - t)^\omega >$

- $\omega$: shape parameter
- cubic if $\omega = 3$
- linear as $\omega \to +\infty$
Shape constraints

- **exponentials** \(<1, t, e^{\omega t}, e^{-\omega t} >
  - \omega : \text{shape parameter}
  - cubic as \(\omega \to 0\)
  - linear as \(\omega \to +\infty\)

- **variable degree** \(<1, t, t^\omega, (1 - t)^\omega >
  - \omega : \text{shape parameter}
  - cubic if \(\omega = 3\)
  - linear as \(\omega \to +\infty\)

- ...
Unifying approach:

**Ex:** $<1, t, u(t), v(t)> \ (\simeq \text{cubics}) \ u, v \in C^2, \ t \in [0, 1]$
Unifying approach: Bernstein-like basis

Ex: $< 1, t, u(t), v(t) > \ (\simeq \text{cubics}) \quad u, v \in C^2, \quad t \in [0, 1]$

ONTP/Bernstein-like basis $\{ B_0, B_1, B_2, B_3 \}$:
Unifying approach: Bernstein-like basis

**Ex:** <1, t, u(t), v(t)> (≃ cubics) u, v ∈ C², t ∈ [0, 1]

**ONTP/Bernstein-like basis** \{B₀, B₁, B₂, B₃\}:

\[
\begin{align*}
B₀(1) &= B₀′(1) = B₀″(1) = 0 \\
B₁(0) &= B₁(1) = B₁′(1) = 0 \\
B₂(0) &= B₂′(0) = B₂(1) = 0 \\
B₃(0) &= B₃′(0) = B₃″(0) = 0
\end{align*}
\]

C² ⇒ easy to characterize/construct
Unifying approach: Bernstein-like basis

- **Ex:** $<1, t, u(t), v(t)> \simeq \text{cubics} \quad u, v \in C^2, \quad t \in [0, 1]$

- **ONTP/Bernstein-like basis** $\{B_0, B_1, B_2, B_3\}$:
  
  \[ B_0(1) = B_0'(1) = B_0''(1) = 0 \]
  \[ B_1(0) = B_1(1) = B_1'(1) = 0 \]
  \[ B_2(0) = B_2'(0) = B_2(1) = 0 \]
  \[ B_3(0) = B_3'(0) = B_3''(0) = 0 \]

  \(C^2 \Rightarrow \) easy to characterize/construct

- **control points:** $(0, b_0), (\xi, b_1), (1 - \eta, b_2), (1, b_3), \ 0 < \xi < 1 - \eta < 1,$
Unifying approach: Bernstein-like basis

Ex: $<1, t, u(t), v(t)> (\simeq \text{cubics})$  $u, v \in C^2$,  $t \in [0, 1]$ 

ONTP/Bernstein-like basis $\{B_0, B_1, B_2, B_3\}$:

- $B_0(1) = B_0'(1) = B_0''(1) = 0$
- $B_1(0) = B_1(1) = B_1'(1) = 0$
- $B_2(0) = B_2'(0) = B_2(1) = 0$
- $B_3(0) = B_3'(0) = B_3''(0) = 0$

$C^2 \Rightarrow$ easy to characterize/construct

control points: $(0, b_0), (\xi, b_1), (1 - \eta, b_2), (1, b_3)$, $0 < \xi < 1 - \eta < 1$

control polygon describes $s(t) = \sum_{j=0}^{3} b_j B_j(t)$
Unifying approach: Bernstein-like basis

- Ex: \( <1, t, u(t), v(t) > (\cong \text{cubics}) \) \( u, v \in C^2, \ t \in [0, 1] \)

- ONTP/Bernstein-like basis \( \{B_0, B_1, B_2, B_3\} \):
  \[
  B_0(1) = B_0'(1) = B_0''(1) = 0 \\
  B_1(0) = B_1(1) = B_1'(1) = 0 \\
  B_2(0) = B_2'(0) = B_2(1) = 0 \\
  B_3(0) = B_3'(0) = B_3''(0) = 0
  \]

- \( C^2 \Rightarrow \text{easy to characterize/construct} \)

- control points: \((0, b_0), (\xi, b_1), (1 - \eta, b_2), (1, b_3), 0 < \xi < 1 - \eta < 1, \)

- control polygon describes \( s(t) = \sum_{j=0}^{3} b_j B_j(t) \)

- properties of \( s \) by its control polygon
Unifying approach:

\[ \mathbb{P}_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle \]
Unifying approach:

\[ \mathcal{P}^{u,v}_p := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle, \ p \geq 2 \ t \in [0, 1] \]
Unifying approach:

\[ \Pi_{p}^{u,v} := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle, \quad p \geq 2 \quad t \in [0, 1] \]

\[ \langle D^{p-1}u, D^{p-1}v \rangle \quad \text{Chebyshev in} \ [0, 1] \ \text{and Extended Chebyshev in} \ (0, 1) \]
Unifying approach: ONTP-basis

\[ \mathbb{P}^u,v_p := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle, \quad p \geq 2 \quad t \in [0, 1] \]

\[ \langle D^{p-1}u, D^{p-1}v \rangle \quad \text{Chebyshev in} \ [0, 1] \text{ and Extended Chebyshev in} \ (0, 1) \]

\[ \downarrow \]

\[ \mathbb{P}^u,v_p \text{ possesses a ONTP-basis} \]
Unifying approach: ONTP-basis

\[ \mathbb{P}^{u,v}_p := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle, \quad p \geq 2 \quad t \in [0, 1] \]

\[ \langle D^{p-1}u, D^{p-1}v \rangle \] Chebyshev in [0, 1] and Extended Chebyshev in (0, 1)

\[ \downarrow \]

\[ \mathbb{P}^{u,v}_p \text{ possesses a ONTP-basis} \]

Ex:
- \( u, v \): trigonometric functions
- \( u, v \): exponential functions
- \( u, v \): variable degree
- ....
Unifying approach: ONTP-basis

\( \Pi_{p}^{u,v} := <1, t, \ldots, t^{p-2}, u(t), v(t)>, \quad p \geq 2 \quad t \in [0, 1] \)

\( <D^{p-1}u, D^{p-1}v> \) Chebyshev in [0, 1] and Extended Chebyshev in (0, 1)

\[\downarrow\]

\( \Pi_{p}^{u,v} \) possesses a ONTP-basis

Bernstein-like representations

[Mainar, E., Peña, J.M., Sánchez-Reyes, J, CAGD 2001]
[Carnicer, Mainar, Peña; CA 2004]
[Mazure, M.-L., CA, 2005]
[Costantini, P., Lyche, T., Manni, C., NM, 2005]

....
Unifying approach: Bernstein-like basis

- smoothness between adjacent segments: easily described by control points
Unifying approach: Bernstein-like basis

- Smoothness between adjacent segments: easily described by control points

\[ C^1 \text{ cubics} \quad \text{and} \quad C^1 \text{ Trig/Exp} \]
Unifying approach: Bernstein-like basis

- smoothness between adjacent segments: easily described by control points

$C^1$ cubics
Unifying approach: Bernstein-like basis

- smoothness between adjacent segments: easily described by control points

\[ C^1 \text{ cubics} \]

\[ C^1 \text{ exponential (cubics)} \]
Spaces good for design

\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle, \quad p \geq 2 \quad t \in [0, 1] \]
Spaces good for design

\[ \mathcal{E} \subset C^n: n + 1 \text{ dimensional EC space containing constants} \]

\[ \mathcal{E} \text{ is Extended Chebyshev (EC) in } I \text{ if any non trivial element has at most } n \text{ zeros in } I \]
Spaces good for design

$\mathcal{E} \subset C^n$: $n + 1$ dimensional EC space containing constants

- $B_0, \cdots, B_n$ is a Bernstein-like basis of $\mathcal{E}$ in $[a, b] \subset I$ if
  - $B_0, \cdots, B_n$ is NTP
  - $B_k$ vanishes exactly $k$ times in $a$ and $n - k$ times in $b$
Spaces good for design

\[ \mathcal{I} \subset C^n: n + 1 \text{ dimensional EC space containing constants} \]

- \( B_0, \cdots B_n \) is a Bernstein-like basis of \( \mathcal{I} \) in \([a, b] \subset I\) if
  - \( B_0, \cdots B_n \) is NTP
  - \( B_k \) vanishes exactly \( k \) times in \( a \) and \( n - k \) times in \( b \)
- A Bernstein-like basis of \( \mathcal{I} \) is the ONTP basis of \( \mathcal{I} \)
Spaces good for design

\[ \mathbb{E} \subset C^n: n + 1 \text{ dimensional EC space containing constants} \]

- \( B_0, \cdots B_n \) is a Bernstein-like basis of \( \mathbb{E} \) in \( [a, b] \subset I \) if
  - \( B_0, \cdots B_n \) is NTP
  - \( B_k \) vanishes exactly \( k \) times in \( a \) and \( n - k \) times in \( b \)

A Bernstein-like basis of \( \mathbb{E} \) is the ONTP basis of \( \mathbb{E} \)

- \( \mathbb{E} \) possesses a Bernstein-like basis in any \( [a, b] \subset I \) iff
  \( \{ f' : f \in \mathbb{E} \} \) is an Extended Chebyshev space in \( I \)
Spaces good for design

\( \mathfrak{E} \subset C^n: n + 1 \) dimensional EC space containing constants

- \( B_0, \ldots B_n \) is a Bernstein-like basis of \( \mathfrak{E} \) in \([a, b] \subset I\) if
  - \( B_0, \ldots B_n \) is NTP
  - \( B_k \) vanishes exactly \( k \) times in \( a \) and \( n - k \) times in \( b \)

- A Bernstein-like basis of \( \mathfrak{E} \) is the ONTP basis of \( \mathfrak{E} \)
- \( \mathfrak{E} \) possesses a Bernstein-like basis in any \([a, b] \subset I\) iff
  \( \{f': f \in \mathfrak{E}\} \) is an Extended Chebyshev space in \( I \)

- in \( \mathfrak{E} \) all classical geometric design algorithms can be developed for the Bernstein-like basis (blossoms)
  \( \Rightarrow \) \( \mathfrak{E} \) is good for design true under less restrictive hypotheses


...
Alternatives to the rational model
Alternatives to the rational model

rational model: \( \mathbb{P}_p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \)
Alternatives to the rational model

- rational model: $\mathbb{P}_p \rightarrow $ B-splines $\rightarrow $ NURBS
- alternative: $\mathbb{P}_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle$

$\downarrow$

$\mathbb{P}^{u,v}_p := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle$
Alternatives to the rational model

- rational model: \( \mathbb{P}_p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \)

- alternative: \( \mathbb{P}_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle \)

\[ \downarrow \]

\[ \mathbb{P}^{u,v}_p := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle \]

- select proper \( \mathbb{P}^{u,v}_p \):
  - good approximation properties
Alternatives to the rational model

- rational model: \( \mathbb{P}_p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \)

- alternative: \( \mathbb{P}_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle \)

\[ \downarrow \]

\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle \]

- select proper \( \mathbb{P}_{p}^{u,v} \):
  - good approximation properties
  - exactly represent salient profiles

\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \ldots, t^{p-2}, \cos \omega t, \sin \omega t \rangle \]

\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \ldots, t^{p-2}, \cosh \omega t, \sinh \omega t \rangle \]
Alternatives to the rational model

- rational model: \( \mathbb{P}_p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \)
- alternative: \( \mathbb{P}_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle \)

\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle \]

- select proper \( \mathbb{P}_{p}^{u,v} \):
  - good approximation properties
  - exactly represent salient profiles

\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \ldots, t^{p-2}, \cos \omega t, \sin \omega t \rangle = \text{TRIG} \]
\[ \mathbb{P}_{p}^{u,v} := \langle 1, t, \ldots, t^{p-2}, \cosh \omega t, \sinh \omega t \rangle = \text{HYP} \]

conic sections, helix, cycloid, ...
Alternatives to the rational model

- **rational model:** \( \mathbb{P}_p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \)
- **alternative:**
  \[
  \mathbb{P}_p = \begin{pmatrix} 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \end{pmatrix}
  \]
  \[
  \mathbb{P}^{u,v}_p := \begin{pmatrix} 1, t, \ldots, t^{p-2}, u(t), v(t) \end{pmatrix}
  \]
  select proper \( \mathbb{P}^{u,v}_p \):
  - good approximation properties
  - describe sharp variations

\[
\begin{align*}
\mathbb{P}^{u,v}_p & := \begin{pmatrix} 1, t, \ldots, t^{p-2}, e^{\omega t}, e^{-\omega t} \end{pmatrix} \\
\mathbb{P}^{u,v}_p & := \begin{pmatrix} 1, t, \ldots, t^{p-2}, (1 - t)^\omega, t^\omega \end{pmatrix}
\end{align*}
\]
Alternatives to the rational model

- rational model: \( \mathbb{P}_p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \)

- alternative: \( \mathbb{P}_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle \)

\[
\downarrow
\]

\( \mathbb{P}_{p}^{u,v} \) := \( \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle \)

- select proper \( \mathbb{P}_{p}^{u,v} \):
  - good approximation properties
  - describe sharp variations

\[
\begin{align*}
\mathbb{P}_{p}^{u,v} & := \langle 1, t, \ldots, t^{p-2}, e^{\omega t}, e^{-\omega t} \rangle = \text{EXP} = (\text{HYP}) \\
\mathbb{P}_{p}^{u,v} & := \langle 1, t, \ldots, t^{p-2}, (1 - t)^{\omega}, t^{\omega} \rangle = \text{VDP}
\end{align*}
\]
 Alternatives to the rational model

- rational model: \( \mathbb{I}^p \rightarrow \text{B-splines} \rightarrow \text{NURBS} \)
- alternative: \( \mathbb{I}^p = < 1, t, \ldots, t^{p-2}, t^{p-1}, t^p > \)
  \[ \downarrow \]
  \( \mathbb{I}^u,v_p := < 1, t, \ldots, t^{p-2}, u(t), v(t) > \)

- construct/analyse spline spaces with sections in \( \mathbb{I}^u,v_p \) with suitable bases for them (analogous to B-splines)

[Lyche, CA 1985]
[Schumaker, L.L.; 1993],
[Koch, P.E, Lyche, T.; Computing 1993],
[Marušic, M., Rogina, M.; JCAM 1995],
[Kvasov, B.I., Sattayatham, P.; JCAM 1999],
[Costantini, P.; CAGD 2000],
[Costantini, P., Manni, C.; RM 2006]
[Wang Fang; JCAM 2008],
Generalized B-splines

\[ \Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\}, \]
\[
\{..., u_i, v_i, ...,\}, < 1, t, \ldots, t^{p-2}, u_i(t), v_i(t)>, < D^{p-1} u_i, D^{p-1} v_i > \text{ Chebyshev}
\]

\[ D^{p-1} v_i(\xi_i) = 0, \quad D^{p-1} v_i(\xi_{i+1}) > 0, \quad D^{p-1} u_i(\xi_i) > 0, \quad D^{p-1} u_i(\xi_{i+1}) = 0, \]
\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1} \}, \]
\[ \{ \ldots, u_i, v_i, \ldots \}, < 1, t, \ldots, t^{p-2}, u_i(t), v_i(t) >, < D^{p-1}u_i, D^{p-1}v_i > \text{ Chebyshev} \]
\[ D^{p-1}v_i(\xi_i) = 0, \quad D^{p-1}v_i(\xi_{i+1}) > 0, \quad D^{p-1}u_i(\xi_i) > 0, \quad D^{p-1}u_i(\xi_{i+1}) = 0, \]
\[ \hat{B}^{(1)}_{i,\Xi}(t) := \begin{cases} 
\frac{D^{p-1}v_i(t)}{D^{p-1}v_i(\xi_{i+1})} & t \in [\xi_i, \xi_{i+1}) \\
\frac{D^{p-1}u_i+1(t)}{D^{p-1}u_i+1(\xi_{i+1})} & t \in [\xi_{i+1}, \xi_{i+2}) \\
0 & \text{elsewhere} 
\end{cases} \]
Generalized B-splines

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1} \}, \]
\[ \{ \ldots, u_i, v_i, \ldots \}, < 1, t, \ldots, t^{p-2}, u_i(t), v_i(t) >, < D^{p-1}u_i, D^{p-1}v_i > \text{ Chebyshev} \]

\[ D^{p-1}v_i(\xi_i) = 0, \quad D^{p-1}v_i(\xi_{i+1}) > 0, \quad D^{p-1}u_i(\xi_i) > 0, \quad D^{p-1}u_i(\xi_{i+1}) = 0, \]

\[ \hat{B}^{(1)}_{i,\Xi}(t) := \begin{cases} 
\frac{D^{p-1}v_i(t)}{D^{p-1}v_i(\xi_{i+1})} & t \in [\xi_i, \xi_{i+1}) \\
\frac{D^{p-1}u_{i+1}(t)}{D^{p-1}u_{i+1}(\xi_{i+1})} & t \in [\xi_{i+1}, \xi_{i+2}) \\
0 & \text{elsewhere} 
\end{cases} \]

\[ \hat{B}^{(p)}_{i,\Xi}(t) = \int_{-\infty}^{t} \hat{\delta}^{(p-1)}_{i,\Xi} \hat{B}^{(p-1)}_{i,\Xi}(s) ds - \int_{-\infty}^{t} \hat{\delta}^{(p-1)}_{i+1,\Xi} \hat{B}^{(p-1)}_{i+1,\Xi}(s) ds \]

\[ \hat{\delta}^{(p)}_{i,\Xi} := \frac{1}{\int_{-\infty}^{+\infty} \hat{B}^{(p)}_{i,W,\Xi}(s) ds} \]
Generalized B-splines

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1} \}, \]
\[ \{ \ldots, u_i, v_i, \ldots \}, \quad < 1, t, \ldots, t^{p-2}, u_i(t), v_i(t) >, < D^{p-1}u_i, D^{p-1}v_i > \text{ Chebyshev} \]

\[ D^{p-1}v_i(\xi_i) = 0, \quad D^{p-1}v_i(\xi_{i+1}) > 0, \quad D^{p-1}u_i(\xi_i) > 0, \quad D^{p-1}u_i(\xi_{i+1}) = 0, \]

\[ \hat{B}^{(1)}_{i,\Xi}(t) := \begin{cases} 
\frac{D^{p-1}v_i(t)}{D^{p-1}v_i(\xi_{i+1})} & t \in [\xi_i, \xi_{i+1}) \\
\frac{D^{p-1}u_{i+1}(t)}{D^{p-1}u_{i+1}(\xi_{i+1})} & t \in [\xi_{i+1}, \xi_{i+2}) \\
0 & \text{elsewhere}
\end{cases} \]

\[ \hat{B}^{(p)}_{i,\Xi}(t) = \int_{-\infty}^{t} \delta^{(p-1)}_{i,\Xi} \hat{B}^{(p-1)}_{i,\Xi}(s)ds - \int_{-\infty}^{t} \delta^{(p-1)}_{i+1,\Xi} \hat{B}^{(p-1)}_{i+1,\Xi}(s)ds \]

\[ \hat{\delta}^{(p)}_{i,\Xi} := \frac{1}{\int_{-\infty}^{+\infty} \hat{B}^{(p)}_{i,W,\Xi}(s)ds} \]

B-splines

\[ B^{(p)}_{i,\Xi}(t) = \int_{-\infty}^{t} \delta^{(p-1)}_{i,\Xi} B^{(p-1)}_{i,\Xi}(s)ds - \int_{-\infty}^{t} \delta^{(p-1)}_{i+1,\Xi} B^{(p-1)}_{i+1,\Xi}(s)ds \]

\[ \delta^{(p)}_{i,\Xi} := \frac{1}{\int_{-\infty}^{+\infty} B^{(p)}_{i,\Xi}(s)ds} \]
Generalized B-splines

$$\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\},$$

$$\{..., u_i, v_i, ...\}, <1, t, \ldots, t^{p-2}, u_i(t), v_i(t)>, <D^{p-1}u_i, D^{p-1}v_i > \text{ Chebyshev}$$

$$D^{p-1}v_i(\xi_i) = 0, \quad D^{p-1}v_i(\xi_{i+1}) > 0, \quad D^{p-1}u_i(\xi_i) > 0, \quad D^{p-1}u_i(\xi_{i+1}) = 0,$$

$$\hat{B}^{(1)}_{i,\Xi}, B^{(1)}_{i,\Xi}$$

All Chebyshevian spline spaces good for design can be built by means of integral recurrence relations, [Mazure M.L., NM 2011]
Generalized B-splines: exponential (hyperbolic)

\[ \Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\} : \text{knots} \quad W := \{..., \omega_i, ...\} : \text{shape parameters} \]

\[ \Pi_p^{u_i,v_i} := <1, t, \ldots, t^{p-2}, \cosh \omega_i t, \sinh \omega_i t> \]

**Exponential case:** \( p = 3 \)

\[ \text{EXP}_3 = \Pi_3^{u,v} := <1, t, e^{\omega t}, e^{-\omega t}> \quad \text{isomorphic to} \quad \Pi_3 \]

Bernstein-like basis

\[ \omega \rightarrow 0: \quad C^2 \text{ cubic B-splines} \]
Generalized B-splines: exponential (hyperbolic)

\[ \Xi := \{ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1} \} : \text{knots} \quad W := \{ ..., \omega_i, ... \} : \text{shape parameters} \]

\[ P_{p}^{u_i,v_i} := \langle 1, t, \ldots, t^{p-2}, \cosh \omega_i t, \sinh \omega_i t \rangle \]

Exponential case: \( p = 3 \)

\[ \text{EXP}_3 = P_{3}^{u,v} := \langle 1, t, e^{\omega t}, e^{-\omega t} \rangle \quad \text{isomorphic to} \quad P_{3} \]

Bernstein-like basis

\[ \omega = 3h \]
Generalized B-splines: properties

\[ \{ \hat{B}_{i, \Xi}^{(p)}(t), \ i = 1, \ldots \} , \]

- Properties analogous to classical B-splines
  - positivity
  - partition of unity: \( p \geq 2 \)
  - compact support
  - smoothness
  - derivatives
  - local linear independence
  - …
Generalized B-splines: properties

\[ \{ \widehat{B}^{(p)}_{i,\Xi}(t), \ i = 1, \ldots \} , \]

- Properties analogous to classical B-splines
  - positivity
  - partition of unity: \( p \geq 2 \)
  - compact support
  - smoothness
  - derivatives
  - local linear independence
  - \ldots
  - shape properties \( \{ \ldots , u_i, v_i, \ldots \} \)
Generalized B-splines: properties

\( \{ \widehat{B}_{i,\Xi}^{(p)}(t), \ i = 1, \ldots \} \),

- Properties analogous to classical B-splines
  - positivity
  - partition of unity: \( p \geq 2 \)
  - compact support
  - smoothness
  - derivatives
  - local linear independence
  - shape properties \( \{ \ldots, u_i, v_i, \ldots \} \)
  - trig. and exp. parts can be mixed
Generalized B-splines: properties

\[ \{ \hat{B}_{i,\Xi}^{(p)}(t), \ i = 1, \ldots \} , \]

- Properties analogous to classical B-splines
  - positivity
  - partition of unity: \( p \geq 2 \)
  - compact support
  - smoothness
  - derivatives
  - local linear independence
  - shape properties \( \{ \ldots , u_i , v_i , \ldots \} \)
  - trig. and exp. parts can be mixed
  - straightforward multivariate extension via tensor product
Summary
\[ P_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle \]

\[ \downarrow \]

\[ P_p^{u,v} := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle \]
Summary

\[ \mathbf{P}_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle \]

\[ \mathbf{P}_p^{u,v} := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle \]

- Bernstein like bases/control polygon
Summary

\[ \mathbb{P}_p = \langle 1, t, \ldots, t^{p-2}, t^{p-1}, t^p \rangle \]

\[ \mathbb{P}^{u,v}_p := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle \]

- Bernstein like bases/control polygon
- Generalized B-splines: spline spaces with sections in \( \mathbb{P}^{u,v}_p \) with suitable bases for them (analogous to B-splines)
Local Refinements

local refinements are crucial in applications (geometric modelling, simulation,...)
Local Refinements

Local refinements are crucial in applications (geometric modelling, simulation,...)
Local Refinements

local refinements are crucial in applications (geometric modelling, simulation,...)
Local Refinements

Local refinements are crucial in applications (geometric modelling, simulation,...)
Local Refinements

Local refinements are crucial in applications (geometric modelling, simulation,...)
DRAWBACKS of tensor product structures

- the tensor product structure prevents local refinements

Alternatives (polynomial B-splines):
DRAWBACKS of tensor product structures

- the tensor product structure **prevents** local refinements

Alternatives (polynomial B-splines):

DRAWBACKS of tensor product structures

- the tensor product structure **prevents** local refinements
  Alternatives (polynomial B-splines):


  - **LR splines** [Dokken T., Lyche T., Pettersen K.F., CAGD 2013],
DRAWBACKS of tensor product structures

- the tensor product structure prevents local refinements
- Alternatives (polynomial B-splines):
  - LR splines [Dokken T., Lyche T., Pettersen K.F., CAGD 2013],
  - Hierarchical bases
DRAWBACKS of tensor product structures

- the tensor product structure prevents local refinements
  Alternatives (polynomial B-splines):
  - LR splines [Dokken T., Lyche T., Pettersen K.F., CAGD 2013],
  - Hierarchical bases
  - Splines over T-meshes
DRAWBACKS of tensor product structures

- the tensor product structure prevents local refinements

Alternatives (polynomial B-splines):


- **LR splines** [Dokken T., Lyche T., Pettersen K.F., CAGD 2013],

- **Hierarchical bases**

- **Splines over T-meshes**

- **B-splines on triangulations**
Generalized Splines: local refinements?

- Generalized splines have \textit{global tensor-product} structure
Generalized Splines: local refinements?

- Generalized splines have **global tensor-product** structure
- some localization techniques can be applied to (some) generalized spline spaces.

  - Hierarchical generalized splines
  - Generalized splines over T-meshes
  - Quadratic Generalized splines over triangulations
Hierarchical model

sequence of $N$ nested tensor-product spline spaces

$V^0 \subset V^1 \subset \cdots \subset V^{N-1}$
Hierarchical B-spline model


sequence of $N$ nested tensor-product spline spaces

$\mathcal{V}^0 \subset \mathcal{V}^1 \subset \cdots \subset \mathcal{V}^{N-1}$

$\mathcal{V}^\ell$ is spanned by a tensor-product B-spline basis $\mathcal{B}^\ell$:

$\mathcal{B}^\ell = \{\ldots, B_{i,\ell}, \ldots\}$
Hierarchical B-spline model

sequence of $N$ nested tensor-product spline spaces

\[ \mathcal{V}^0 \subset \mathcal{V}^1 \subset \cdots \subset \mathcal{V}^{N-1} \]

$\mathcal{V}^\ell$ is spanned by a tensor-product B-spline basis $\mathcal{B}^\ell$:

\[ \mathcal{B}^\ell = \{ \ldots, B_{i,\ell}, \ldots \} \]

sequence of $N$ nested domains

\[ \Omega_{N-1} \subset \Omega_{N-2} \subset \cdots \subset \Omega_0, \quad \Omega_N = \emptyset \]
Hierarchical B-spline model

Recursive definition

(I) Initialization: $\mathcal{H}^0 := \mathcal{B}^0$
Hierarchical B-spline model

Recursive definition

(I) Initialization: $\mathcal{H}^0 := \mathcal{B}^0$

(II) Construction of $\mathcal{H}^{\ell+1}$ from $\mathcal{H}^{\ell}$,

$\mathcal{H}^{\ell+1} := \mathcal{H}^{\ell+1}_C \cup \mathcal{H}^{\ell+1}_F$

$\ell = 0, 1, \ldots, N - 1$

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]
Hierarchical B-spline model

Recursive definition

(I) Initialization: $\mathcal{H}^0 := \mathcal{B}^0$

(II) construction of $\mathcal{H}^{\ell+1}$ from $\mathcal{H}^\ell$, $\ell = 0, 1, \ldots, N - 1$

$\mathcal{H}^{\ell+1} := \mathcal{H}^{\ell+1}_C \cup \mathcal{H}^{\ell+1}_F$

$\mathcal{H}^{\ell+1}_C := \{ B_{i,\ell} \in \mathcal{H}^{\ell} : \text{supp}(B_{i,\ell}) \not\subset \Omega_{\ell+1} \}$

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]
Hierarchical B-spline model

Recursive definition

(I) Initialization: $\mathcal{H}^0 := \mathcal{B}^0$

(II) construction of $\mathcal{H}^{\ell+1}$ from $\mathcal{H}^\ell$,

$$\mathcal{H}^{\ell+1} := \mathcal{H}_C^{\ell+1} \cup \mathcal{H}_F^{\ell+1}$$

$\ell = 0, 1, \ldots, N - 1$

$$\mathcal{H}_C^{\ell+1} := \{ B_{i,\ell} \in \mathcal{H}^\ell : \text{supp}(B_{i,\ell}) \not\subset \Omega_{\ell+1} \}$$

$$\mathcal{H}_F^{\ell+1} := \{ B_{i,\ell+1} \in \mathcal{B}^{\ell+1} : \text{supp}(B_{i,\ell+1}) \subset \Omega_{\ell+1} \}$$

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]
Hierarchical B-spline model

- sequence of $N$ nested tensor-product spline spaces
  \[ \mathcal{V}^0 \subset \mathcal{V}^1 \subset \ldots \subset \mathcal{V}^{N-1} \]
  \( \mathcal{V}^\ell \) is spanned by a tensor-product B-spline basis \( \mathcal{B}^\ell \):
  \[ \mathcal{B}^\ell = \{ \ldots, B_{i,\ell}, \ldots \} \]

- sequence of $N$ nested domains
  \[ \Omega_{N-1} \subset \Omega_{N-2} \subset \ldots \subset \Omega_0, \quad \Omega_N = \emptyset \]
Hierarchical Generalized B-spline model

Generalized B-splines support a hierarchical refinement

- sequence of $N$ nested tensor-product spline spaces
  \[ V^0 \subset V^1 \subset \cdots \subset V^{N-1} \]

- $V^\ell$ spanned by a tensor-product Generalized B-spline basis $\widehat{B}^\ell$:
  \[ \widehat{B}^\ell = \{ \ldots, \widehat{B}_{i,\ell}, \ldots \} \]

- sequence of $N$ nested domains
  \[ \Omega_{N-1} \subset \Omega_{N-2} \subset \cdots \subset \Omega_0, \quad \Omega_N = \emptyset \]
Hierarchical Generalized B-spline model

Generalized B-splines support a hierarchical refinement

- sequence of $N$ nested tensor-product spline spaces

\[ \mathcal{V}^0 \subset \mathcal{V}^1 \subset \cdots \subset \mathcal{V}^{N-1} \]

$\mathcal{V}^\ell$ spanned by a tensor-product Generalized B-spline basis $\widehat{\mathcal{B}}^\ell$:

\[ \widehat{\mathcal{B}}^\ell = \{ \ldots, \widehat{B}_{i,\ell}, \ldots \} \]

- sequence of $N$ nested domains

\[ \Omega_{N-1} \subset \Omega_{N-2} \subset \cdots \subset \Omega_0, \quad \Omega_N = \emptyset \]

⇒ similar recursive definition
Hierarchical B-splines model

1D Example: Cubic B-spline basis

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]
Hierarchical B-splines model

1D Example: Cubic B-spline basis

\[ H^0 = B^0 \]

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]
Hierarchical B-splines model

1D Example: Cubic B-spline basis

\[ \mathcal{H}^0 = \mathcal{B}^0 \]

\[ \mathcal{B}^1 \]

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]
Hierarchical B-splines model

1D Example: Cubic B-spline basis

\[ \mathcal{H}_C^1 \cup \mathcal{H}_F^1 \]

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]
Hierarchical B-splines model

1D Example: Cubic B-spline basis

\[ \mathcal{H}^1 \]

\[ \mathcal{H}^1_{C} \]

\[ \bigcup \]

\[ \mathcal{H}^1_{F} \]

\[ \downarrow \]

\[ \mathcal{H}^1 \]

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]
Hierarchical Generalized B-spline model

Generalized B-splines support a hierarchical refinement

1D Example: \( \text{EXP}_3 \) B-splines basis \( \omega_i = 50 \)
Hierarchical **Generalized B-spline model**

Generalized B-splines support a hierarchical refinement

1D Example: $\text{EXP}_3$ B-splines basis $\omega_i = 50$

![Diagram of hierarchical refinement with EXP3 B-splines basis](image)
Hierarchical Generalized B-spline model

The main properties of Hierarchical B-splines are inherited by Hierarchical GB-splines
Hierarchical Generalized B-spline model

The main properties of Hierarchical B-splines are inherited by Hierarchical GB-splines

- the functions in $\mathcal{H}^\ell$ obtained by the iterative procedure are linearly independent
The main properties of Hierarchical B-splines are inherited by Hierarchical GB-splines

- the functions in $\mathcal{H}^\ell$ obtained by the iterative procedure are linearly independent
- the hierarchical bases $\mathcal{H}^\ell$, for each $\ell$, span nested spaces:

$$\text{span}\mathcal{H}^\ell \subseteq \text{span}\mathcal{H}^{\ell+1}$$
Hierarchical Generalized B-spline model

The main properties of Hierarchical B-splines are inherited by Hierarchical GB-splines

- the functions in $\mathcal{H}^\ell$ obtained by the iterative procedure are linearly independent
- the hierarchical bases $\mathcal{H}^\ell$, for each $\ell$, span nested spaces:

$$\text{span}\mathcal{H}^\ell \subseteq \text{span}\mathcal{H}^{\ell+1}$$

- positivity
Hierarchical Generalized B-spline model

The main properties of Hierarchical B-splines are inherited by Hierarchical GB-splines

- the functions in $\mathcal{H}^\ell$ obtained by the iterative procedure are linearly independent
- the hierarchical bases $\mathcal{H}^\ell$, for each $\ell$, span nested spaces:

$$\text{span} \mathcal{H}^\ell \subseteq \text{span} \mathcal{H}^{\ell+1}$$

- positivity
- partition of unity
Hierarchical Generalized B-spline model

The main properties of Hierarchical B-splines are inherited by Hierarchical GB-splines

- the functions in $H^\ell$ obtained by the iterative procedure are linearly independent
- the hierarchical bases $H^\ell$, for each $\ell$, span nested spaces:
  $$\text{span} H^\ell \subseteq \text{span} H^{\ell+1}$$

- positivity
- partition of unity
  - by using truncated bases
    [Giannelli, Jüttler, Speleers; AiCM 2013 ]
Hierarchical Generalized B-spline model

Generalized B-splines: truncated hierarchical basis

1D Example: $\text{EXP}_3$ B-splines basis $\omega_i = 50$
Hierarchical Generalized B-spline model

Generalized B-splines: truncated hierarchical basis

1D Example: $\text{EXP}_3$ B-splines basis $\omega_i = 50$

$\mathcal{H}^0$

$\downarrow$

$\mathcal{H}^1$
Hierarchical Generalized B-spline model

Generalized B-splines: truncated hierarchical basis

1D Example: $\text{EXP}_3$ B-splines basis $\omega_i = 50$

\[
\begin{align*}
\text{T}^0 \\
\downarrow \\
\text{T}^1
\end{align*}
\]
sequence of $N$ nested tensor-product spline spaces

$V^0 \subset V^1 \subset \ldots \subset V^{N-1}$
Hierarchical Generalized B-splines: space

- sequence of $N$ nested tensor-product spline spaces

$$\mathcal{V}^0 \subset \mathcal{V}^1 \subset \ldots \subset \mathcal{V}^{N-1}$$

- $\mathcal{V}^\ell$ tensor-product (Generalized) B-splines

- sequence of $N$ nested domains

$$\Omega_{N-1} \subset \Omega_{N-2} \subset \ldots \subset \Omega_0, \quad \Omega_N = \emptyset$$
Hierarchical Generalized B-splines: space

- sequence of $N$ nested tensor-product spline spaces
  \[ V^0 \subset V^1 \subset \cdots \subset V^{N-1} \]

- $V^\ell$ tensor-product (Generalized) B-splines

- sequence of $N$ nested domains
  \[ \Omega_{N-1} \subset \Omega_{N-2} \subset \cdots \subset \Omega_0, \quad \Omega_N = \emptyset \]

hierarchical (Generalized) B-splines span the full space

\[ \{ f : f|_{\Omega_0 \setminus \Omega_{\ell+1}} \in V^\ell|_{\Omega_0 \setminus \Omega_{\ell+1}}, \ell = 0, \ldots, N - 1 \} \]

[Giannelli, Jüttler; JCAM 2013], [Speleers, Manni, 2013 preprint]
Hierarchical structures: not nested spaces

the construction can be applied to a hierarchy of not nested spaces

\( \mathcal{V}^0, \mathcal{V}^1, \ldots, \mathcal{V}^{N-1} \)
Hierarchical structures: not nested spaces

the construction can be applied to a hierarchy of not nested spaces

\( \mathbb{V}^0, \mathbb{V}^1, \ldots, \mathbb{V}^{N-1} \)
Hierarchical structures: not nested spaces

the construction can be applied to a hierarchy of not nested spaces

\[ \mathbb{V}^0, \mathbb{V}^1, \ldots, \mathbb{V}^{N-1} \]

great flexibility
Hierarchical structures: not nested spaces

the construction can be applied to a hierarchy of not nested spaces

\[ V^0, V^1, \ldots, V^{N-1} \]

- great flexibility
- different section spaces at different levels
Hierarchical structures: not nested spaces

the construction can be applied to a hierarchy of not nested spaces

\[ \mathcal{V}^0, \mathcal{V}^1, \ldots, \mathcal{V}^{N-1} \]

great flexibility

different section spaces at different levels

the functions in \( \mathcal{H}^\ell \) obtained by the iterative procedure remain linearly independent
Hierarchical structures: not nested spaces

the construction can be applied to a hierarchy of not nested spaces

\[ \mathbb{V}_0, \mathbb{V}_1, \ldots, \mathbb{V}_{N-1} \]

- great flexibility
- different section spaces at different levels
- the functions in \( \mathcal{H}^\ell \) obtained by the iterative procedure remain linearly independent
- not nested spaces \( \text{span} \mathcal{H}^\ell \)

[Manni, Pelosi, Speleers; 2013, to appear]
Hierarchical B-splines are particular bases of particular spline spaces on special rectangular partitions
Spline spaces over T-meshes
Spline spaces over T-meshes

\[ T \text{-mesh } \mathcal{T} \]

partition of a (rectangular) domain by rectangles: T-junctions (hanging vertices) are allowed
Spline spaces over T-meshes

T-mesh $\mathcal{T}$

Partition of a (rectangular) domain by rectangles: **T-junctions** (hanging vertices) are allowed

$$S^r_d(\mathcal{T}) := \{ s(x, y) \in C^r, \ s(x, y)|_{\tau_i} \in \mathbb{P}_{d_1} \times \mathbb{P}_{d_2}, \ \tau_i \in \mathcal{T} \},$$

$$\mathbb{P}_d := \left\{ q(z) = \sum_{j=0}^{d} z^j \right\}, \ \mathbf{r} = (r_1, r_2), \ \mathbf{d} = (d_1, d_2)$$

[Deng, J.-S., Chen, F.-L., Feng, Y.-Y., JCAM 2006]
[Schumaker, L. L. and Wang, L., CAGD 2012]
[Schumaker, L. L. and Wang, L., NM 2011]
Spline spaces over T-meshes

T-mesh $\mathcal{T}$

Partition of a (rectangular) domain by rectangles: T-junctions (hanging vertices) are allowed

$$S^r_d(\mathcal{T}) := \left\{ s(x, y) \in C^r, \ s(x, y)|_{\tau_i} \in \mathbb{P}_{d_1} \times \mathbb{P}_{d_2}, \ \tau_i \in \mathcal{T} \right\},$$

$$\mathbb{P}_d := \left\{ q(z) = \sum_{j=0}^{d} z^j \right\}, \ r = (r_1, r_2), \ d = (d_1, d_2)$$

Polynomial reproduction

$\sim$ Dimension?

$\sim$ Suitable bases?
Spline spaces over T-meshes: dimension

\[ \dim(\mathcal{S}_d^r(\mathcal{T})) = F(d_1 + 1)(d_2 + 1) - E_h(d_2 + 1)(r_2 + 1) - E_v(d_1 + 1)(r_1 + 1) + V(r_1 + 1)(r_2 + 1) \]

+ homology term

\( F : \#faces, \ E_h : \#hor.edges, \ E_v : \#vert.edges, \ V : \#int.vertices \)
Spline spaces over T-meshes: dimension

\[ \dim(S^r_d(T)) = \]
\[ F(d_1 + 1)(d_2 + 1) - E_h(d_2 + 1)(r_2 + 1) - E_v(d_1 + 1)(r_1 + 1) + V(r_1 + 1)(r_2 + 1) + \text{ homology term} \]

\[ F : \# \text{faces}, \ E_h : \# \text{hor.edges}, \ E_v : \# \text{vert.edges}, \ V : \# \text{int.vertices} \]

\[ d \geq 2r + 1, \]
\[ \dim(S^r_d(T)) = \]
\[ F(d_1 + 1)(d_2 + 1) - E_h(d_2 + 1)(r_2 + 1) - E_v(d_1 + 1)(r_1 + 1) + V(r_1 + 1)(r_2 + 1) \]

[Deng, J.-S., Chen, F.-L., Feng, Y.-Y., JCAM 2006]
[Schumaker, L. L. and Wang, L., 2011, CAGD 2012]
[Schumaker, L. L. and Wang, L., NM 2011]
Spline spaces over T-meshes: dimension

\[ \text{dim}(S^r_d(T)) = \]
\[ F(d_1 + 1)(d_2 + 1) - E_h(d_2 + 1)(r_2 + 1) - E_v(d_1 + 1)(r_1 + 1) + V(r_1 + 1)(r_2 + 1) \]
\[ + \text{homology term} \]
\[ F : \# \text{faces}, \ E_h : \# \text{hor. edges}, \ E_v : \# \text{vert. edges}, \ V : \# \text{int. vertices} \]

\[ d \geq 2r + 1, \]
\[ \text{dim}(S^r_d(T)) = \]
\[ F(d_1 + 1)(d_2 + 1) - E_h(d_2 + 1)(r_2 + 1) - E_v(d_1 + 1)(r_1 + 1) + V(r_1 + 1)(r_2 + 1) \]

\[ C^1 \text{ cubics: } \text{dim}(S^1_3(T)) = 4(V_b + V_+) \]
\[ V_b : \# \text{b. vertices}, \ V_+ : \# \text{cross. vertices} \]
\[ \text{Ex: } \text{dim}(S^1_3(T)) = 4(9 + 1) \]
Splines over T-meshes: dimension

\[ d \geq 2r + 1, \] rectangular domains: results based on

- Bernstein representation
- minimal determining sets

[Alfeld, P., Schumaker, L.L., CA 1987]
[Alfeld P., JCAM 2000]
[Deng, J.-S., Chen, F-L., Feng, Y.-Y., JCAM 2006]
[Schumaker, L. L. and Wang, L., 2011, preprint]
[Schumaker, L. L. and Wang, L., NM 2011]
Splines over T-meshes: dimension

- $d \geq 2r + 1$, rectangular domains: results based on
  - Bernstein representation
  - minimal determining sets

[Alfeld, P., Schumaker, L.L., CA 1987]
[Alfeld P., JCAM 2000]
[Deng, J.-S., Chen, F-L., Feng, Y.-Y., JCAM 2006]
[Schumaker, L. L. and Wang, L., 2011, preprint]
[Schumaker, L. L. and Wang, L., NM 2011]

- smoothing cofactors
  [Wang, R.-H., 2001]
  [Huang, Z.-J., Deng J.-S. Feng, Y.-Y., Chen, F.-L., JCM 2006]
Generalized Splines over T-meshes

T-mesh: \( \mathcal{T} \)

partition of a (rectangular) domain by rectangles

so that T-junctions (hanging vertices) are allowed
Generalized Splines over T-meshes

T-mesh: $\mathcal{T}$
partition of a (rectangular) domain by rectangles
so that T-junctions (hanging vertices) are allowed

\[ \hat{S}_d^r(\mathcal{T}) := \{ s(x, y) \in C^r, \ s(x, y)|_{\tau_i} \in \mathbb{P}^{u_1, v_1}_{d_1} \otimes \mathbb{P}^{u_2, v_2}_{d_2}, \ \tau_i \in \mathcal{T} \}, \]

\[ \mathbb{P}^{u,v}_p := \langle 1, t, \ldots, t^{p-2}, u(t), v(t) \rangle \]
Generalized Splines over T-meshes

- suitable spaces: exponential, trigonometric
Generalized Splines over T-meshes

- suitable spaces: exponential, trigonometric
- smoothness cond.: Bernstein like representation
Generalized Splines over T-meshes

- suitable spaces: exponential, trigonometric
- smoothness cond.: Bernstein like representation

$C^1$ cubics
Generalized Splines over T-meshes

- suitable spaces: exponential, trigonometric
- smoothness cond.: Bernstein like representation

$C^1$ cubics

$C^1$ exponential (cubics)
Generalized Splines over T-meshes

$C^1$ cubics
Generalized Splines over T-meshes

$C^1$ cubics

$C^1$ trigonometric (cubics), $\omega = \frac{2}{3} \pi$
Generalized Splines over T-meshes: dimension

- trigonometric/exponential $C^1$ cubics:

\[ \dim(\hat{S}_3^1(T)) = 4(V_b + V_+) \]

$V_b$: \#b. vertices, $V_+$: \#cross. vertices

\[ \dim(S_3^1(T)) = 4(9 + 1) \]
Hierarchical bases, T-meshes: similar behavior of B-splines/GB-splines
Hierarchical bases, T-meshes: similar behavior of B-splines/GB-splines

Triangulations?
Quadratic Generalized Splines over Triangles
Quadratic Generalized Splines over Triangles

\[ P_{2}^{u,v} := \langle 1, u(t), v(t) \rangle \]
Quadratic Generalized Splines over Triangles

- $P_{2}^{u,v} := \langle 1, u(t), v(t) \rangle$

- ONTP basis $\{ B_0, B_1, B_2 \}$  $B_0(0) = 1$, $B_0(1) = B_0'(1) = 0$, ...
Quadratic Generalized Splines over Triangles

1. $\mathbb{P}_{2}^{u,v} := \langle 1, u(t), v(t) \rangle$

2. ONTP basis $\{B_0, B_1, B_2\}$  $B_0(0) = 1$, $B_0(1) = B'_0(1) = 0$, $\ldots$

3. Bernstein like representation
   control polygon for functions?

   $$ t \notin < 1, u(t), v(t) > $$

No Greville abscissae
Quadratic Generalized Splines over Triangles

\[ \mathbb{P}^{u,v}_2 := \langle 1, u(t), v(t) \rangle \]

- ONTP basis \( \{B_0, B_1, B_2\} \) \( B_0(0) = 1, \; B_0(1) = B'_0(1) = 0, \cdots \)
- control points \( f = b_0 B_0 + b_1 B_1 + b_2 B_2 \in \mathbb{P}^{u,v}_2 \)

\[ \downarrow \]

\[ (0, b_0), (\xi, b_1), (1 - \xi, b_1), (1, b_2) \quad B_0(t) = B_2(1 - t) \quad \xi = -1/B'_0(0) = 1/B'_2(1) \]
Quadratic Generalized Splines over Triangles

- \( P_{u,v}^2 := < 1, u(t), v(t) > \)

- ONTP basis \( \{ B_0, B_1, B_2 \} \) \( B_0(0) = 1, B_0(1) = B_0'(1) = 0, \cdots \)

- control points \( f = b_0 B_0 + b_1 B_1 + b_2 B_2 \in P_{u,v}^2 \)

\[
\downarrow
\]

\((0, b_0), (\xi, b_1), (1 - \xi, b_1), (1, b_2) \quad B_0(t) = B_2(1 - t) \quad \xi = -1/B_0'(0) = 1/B_2'(1)\]
Quadratic Generalized Splines over Triangles

- \( \mathbb{P}^{u,v}_2 := \langle 1, u(t), v(t) \rangle \)
- ONTP basis \( \{B_0, B_1, B_2\} \) \( B_0(0) = 1, \ B_0(1) = B'_0(1) = 0, \cdots \)
- control points \( f = b_0 B_0 + b_1 B_1 + b_2 B_2 \in \mathbb{P}^{u,v}_2 \)

\[ (0, b_0), (\xi, b_1), (1 - \xi, b_1), (1, b_2) \quad B_0(t) = B_2(1 - t) \quad \xi = -1/B'_0(0) = 1/B'_2(1) \]

- geometric properties of the usual control polygon
Quadratic Generalized Splines over Triangles

\[ H_\omega := < 1, \cosh \omega t, \sinh \omega t >, \ t \in [0, 1] \]
Quadratic Generalized Splines over Triangles

\[ \mathbb{H}_\omega := <1, \cosh \omega t, \sinh \omega t>, \quad t \in [0, 1] \]

\( \omega = 0.1 \)  \( \omega = 1.5 \)  \( \omega = 10 \)
Quadratic Generalized Splines over Triangles

\[ \mathbb{H}_\omega := \langle 1, \cosh \omega t, \sinh \omega t \rangle, \quad t \in [0, 1] \]

\( \omega = 0.1 \)  \hspace{1cm} \omega = 1.5  \hspace{1cm} \omega = 10

ONTP basis \( B_{0,\omega}, B_{1,\omega}, B_{2,\omega}, \omega \to 0 \) quadratic Bernstein pol.
Quadratic Generalized Splines over Triangles

\[ H_\omega := <1, \cosh \omega t, \sinh \omega t>, \quad t \in [0, 1] \]
Quadratic Generalized Splines over Triangles

\[ H_\omega := \langle 1, \cosh \omega t, \sinh \omega t \rangle, \ t \in [0, 1] \]
Quadratic Generalized Splines over Triangles

\[ \mathbb{H}_\omega := \langle 1, \cosh \omega t, \sinh \omega t \rangle, \quad t \in [0, 1] \]
Quadratic Generalized Splines over Triangles

\[ \mathbb{H}_\omega := \langle 1, \cosh \omega t, \sinh \omega t \rangle, \quad t \in [0, 1] \]
Quadratic Generalized Splines over Triangles

\[ X = \tau_1 V_1 + \tau_2 V_2 + \tau_3 V_3 \]
Quadratic Generalized Splines over Triangles

\[ \mathbf{X} = \tau_1 \mathbf{V}_1 + \tau_2 \mathbf{V}_2 + \tau_3 \mathbf{V}_3 \]

\[ < 1, \tau_1, \tau_2, \tau_3 = 1 - \tau_1 - \tau_2, \tau_1^2, \tau_2^2, \tau_3^2 >, \]
Quadratic Generalized Splines over Triangles

\[ X = \tau_1 V_1 + \tau_2 V_2 + \tau_3 V_3 \]

\[ H_\omega := < 1, \cosh \omega \tau_1, \sinh \omega \tau_1, \cosh \omega \tau_2, \sinh \omega \tau_2, \cosh \omega \tau_3, \sinh \omega \tau_3 >, \]
Quadratic Generalized Splines over Triangles

$$X = \tau_1 V_1 + \tau_2 V_2 + \tau_3 V_3$$

$$\mathbb{H}_\omega := < 1, \cosh \omega \tau_1, \sinh \omega \tau_1, \cosh \omega \tau_2, \sinh \omega \tau_2, \cosh \omega \tau_3, \sinh \omega \tau_3 >,$$

$$\dim(\mathbb{H}_\omega) = 7$$
Quadratic Generalized Splines over Triangles

\[ \mathbf{X} = \tau_1 \mathbf{V}_1 + \tau_2 \mathbf{V}_2 + \tau_3 \mathbf{V}_3 \]

\[ \mathbb{H}_\omega := < 1, \cosh \omega \tau_1, \sinh \omega \tau_1, \cosh \omega \tau_2, \sinh \omega \tau_2, \cosh \omega \tau_3, \sinh \omega \tau_3 >, \]

\[ \mathbb{H}_\omega|_{\tau_3=0} := < 1, \cosh \omega \tau_1, \sinh \omega \tau_1 >, \]
Quadratic Generalized Splines over Triangles

\[ \mathbf{X} = \tau_1 \mathbf{V}_1 + \tau_2 \mathbf{V}_2 + \tau_3 \mathbf{V}_3 \]

\[ B_{200,\omega}(\mathbf{X}) = B_{2,\omega}(\tau_1), \]

\[ B_{020,\omega}(\mathbf{X}) = B_{2,\omega}(\tau_2), \]

\[ B_{002,\omega}(\mathbf{X}) = B_{2,\omega}(\tau_3) \]
Quadratic Generalized Splines over Triangles

\[ B_{200,\omega} \quad B_{020,\omega} \quad B_{002,\omega} \]

\[ \omega = 0.1 \]
Quadratic Generalized Splines over Triangles

$B_{200, \omega}$  $B_{020, \omega}$  $B_{002, \omega}$

$\omega = 10$
Quadratic Generalized Splines over Triangles

\[ B_{110,\omega} \]
Quadratic Generalized Splines over Triangles

\[ B_{110,\omega} \]
$B_{110,\omega}$

7 suitable interp. conditions to recover edge behavior
Quadratic Generalized Splines over Triangles

\[ B_{110,\omega} \]

7 suitable interp. conditions to recover edge behavior

easy: 6 function values at *
Quadratic Generalized Splines over Triangles

\[ B_{110, \omega} \]

7 suitable interp. conditions to recover edge behavior

- easy: 6 function values at \(*\)
- exotic: second derivative at one vertex to mimic the polynomial case
Quadratic Generalized Splines over Triangles

\[ B_{110}, \omega \quad B_{101}, \omega \quad B_{011}, \omega \]

\[ \omega = 0.1 \]
Quadratic Generalized Splines over Triangles

\[ B_{110}, \omega \]  
\[ B_{101}, \omega \]  
\[ B_{011}, \omega \]

\( \omega = 10 \)
Quadratic Generalized Splines over Triangles

one function still missed

\[ B_{111,\omega} \]
$B_{111,\omega} = 1 - \sum_{i+j+k=2} B_{ijk,\omega}$
Quadratic Generalized Splines over Triangles

\[ B_{111,\omega} = 1 - \sum_{i+j+k=2} B_{ijk,\omega} \]

\[ \omega = .1 \]
Quadratic Generalized Splines over Triangles

\[ B_{111,\omega} = 1 - \sum_{i+j+k=2} B_{ijk,\omega} \]

\[ \omega = 10 \]
Quadratic Generalized Splines over Triangles

- \( B_{ijk, \omega} \geq 0 \)
- partition of unity
Quadratic Generalized Splines over Triangles
Quadratic Generalized Splines over Triangles
Quadratic G. Splines over Triangles: control net.

NO Greville abscissae
Quadratic G. Splines over Triangles: control net.

\[ \omega = 0.01 \quad \quad \omega = 1.5 \]
Quadratic G. Splines over Triangles: control net.

\[ \omega = 1.5 \]

\[ \omega = 2.57 \]
Quadratic G. Splines over Triangles: control net.

\[ \omega = 2.57 \quad \omega = 3 \]
Quadratic G. Splines over Triangles: control net.

$\omega = 3$

$\omega = 10$
Quadratic G. Splines over Triangles: Smoothness

USUAL geometric interpretation
Quadratic G. Splines over Triangles: Smoothness

USUAL geometric interpretation

\[ \omega = 0.1 \]
USUAL geometric interpretation

\[ \omega = 1.5 \]
Quadratic G. Splines over Triangles: Smoothness

USUAL geometric interpretation

\[ \omega = 10 \]
Conclusions

Bernstein-like representations
- optimal from geometrical and computational point of view
- not confined to (piecewise) polynomial spaces
Conclusions

- Bernstein-like representations
  - optimal from geometrical and computational point of view
  - not confined to (piecewise) polynomial spaces

- Generalized (trigonometric/exponential/...) B-splines
  possible alternative to the rational model

- Bernstein-like representations
- CAGD applications
- IgA applications
Conclusions

- Bernstein-like representations
  - optimal from geometrical and computational point of view
  - not confined to (piecewise) polynomial spaces

- Generalized (trigonometric/exponential/...) B-splines
  possible alternative to the rational model
  - Bernstein-like representations
  - CAGD applications
  - IgA applications

- Local refinements B-splines/Generalized B-splines
  - Hierarchical bases
  - T-meshes
Conclusions

- Bernstein-like representations
  - optimal from geometrical and computational point of view
  - not confined to (piecewise) polynomial spaces
- Generalized (trigonometric/exponential/...) B-splines possible alternative to the rational model
  - Bernstein-like representations
  - CAGD applications
  - IgA applications
- Local refinements B-splines/Generalized B-splines
  - Hierarchical bases
  - T-meshes
- B-splines and GB-splines similar structure/properties thanks to 1D Bernstein-like representation.
Conclusions

- Bernstein-like representations
  - optimal from geometrical and computational point of view
  - not confined to (piecewise) polynomial spaces

- Generalized (trigonometric/exponential/...) B-splines possible alternative to the rational model
  - Bernstein-like representations
  - CAGD applications
  - IgA applications

- Local refinements B-splines/Generalized B-splines
  - Hierarchical bases
  - T-meshes

- B-splines and GB-splines similar structure/properties thanks to 1D Bernstein-like representation.

- Extending Bernstein representations/Generalized B-splines to triangles is not trivial
Many Thanks!