Generalized B-splines and local refinements

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collaboration with

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Outline

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- Bernstein-like representations

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- Generalized B-splines

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Ariadne's thread

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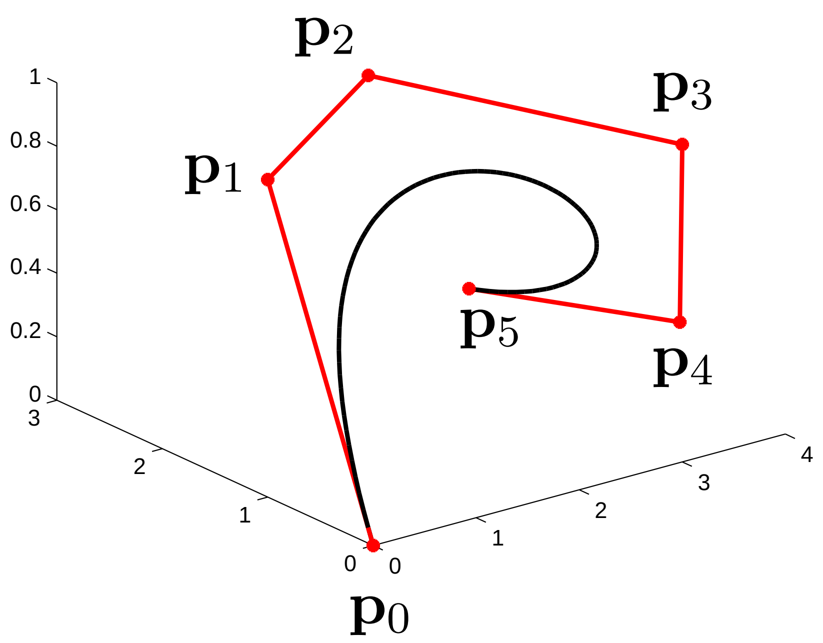
CAGD: Bézier forms

$$\sum_{i=0}^{p} \mathbf{p}_i \binom{p}{i} t^i (1-t)^{p-i}, \qquad t \in [0,1],\ \mathbf{p}_i \in \mathbb{R}^d$$



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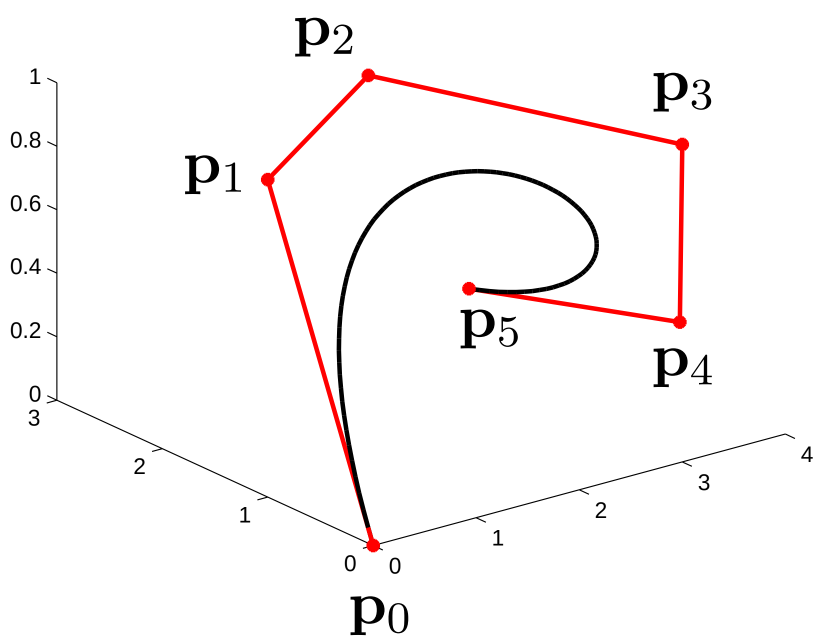

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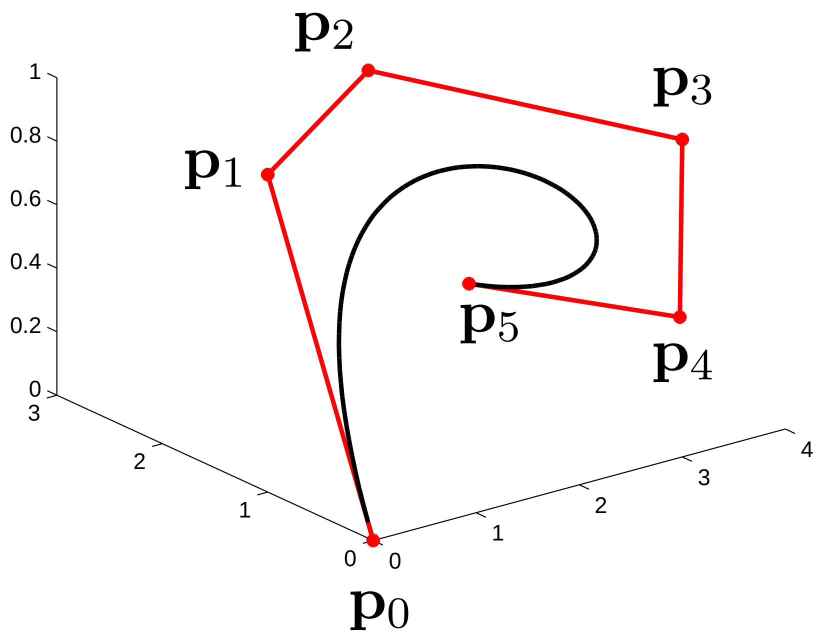

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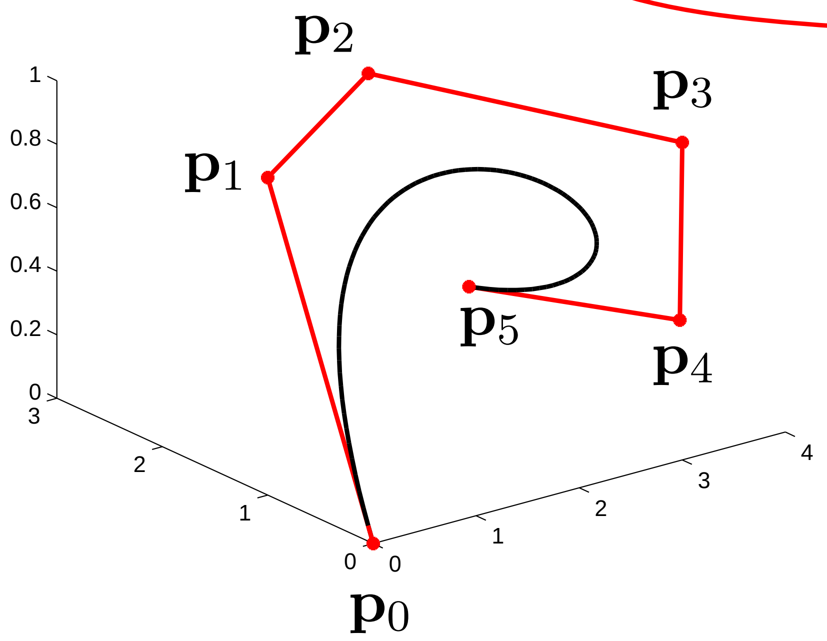

space: $\mathbb{P}_p$

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BÉZIER CURVE

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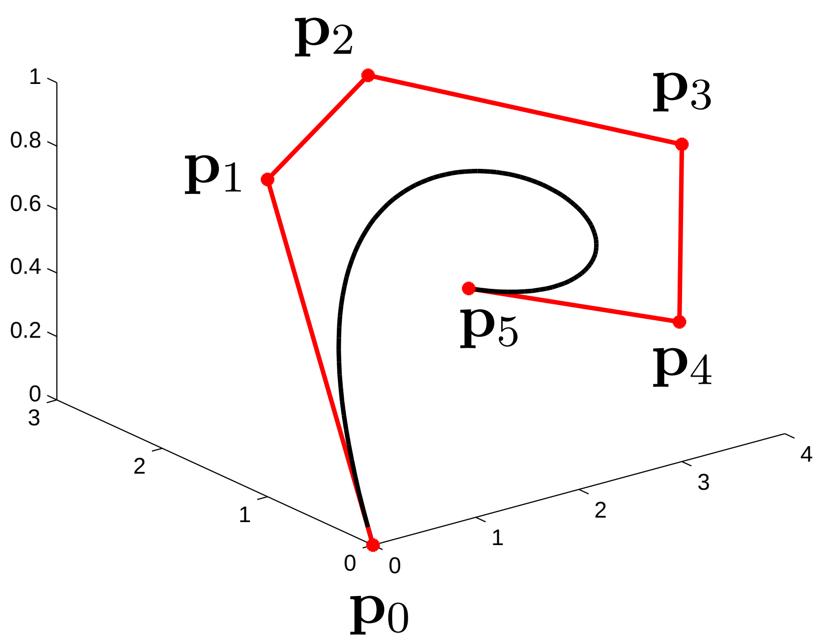

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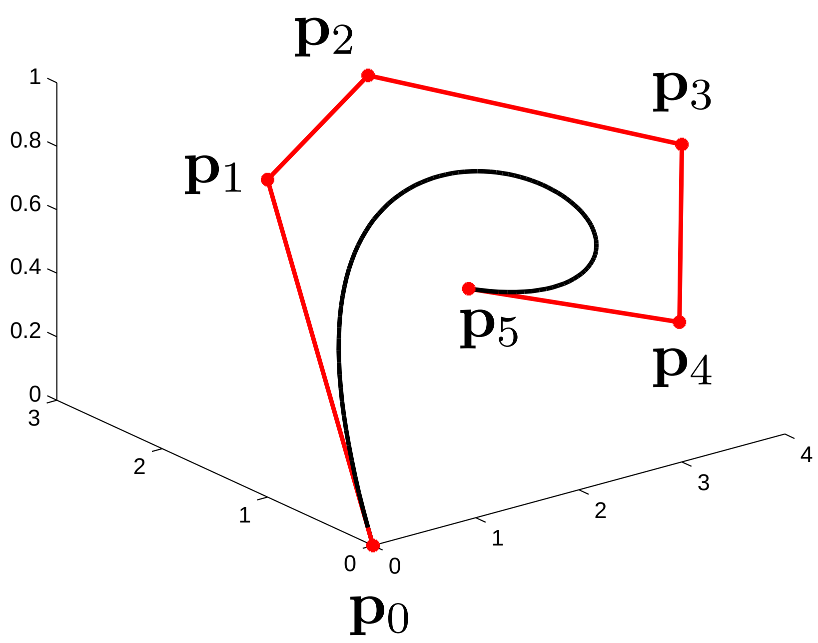

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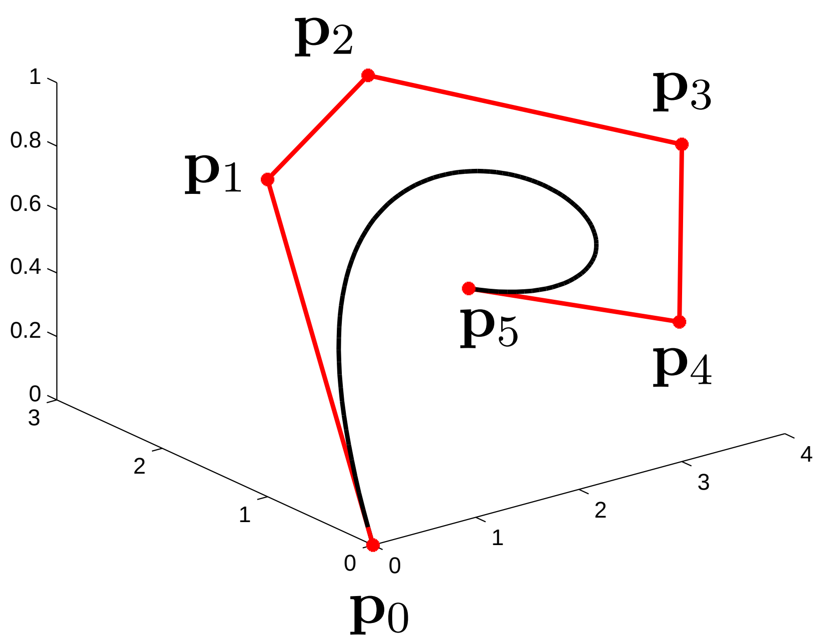

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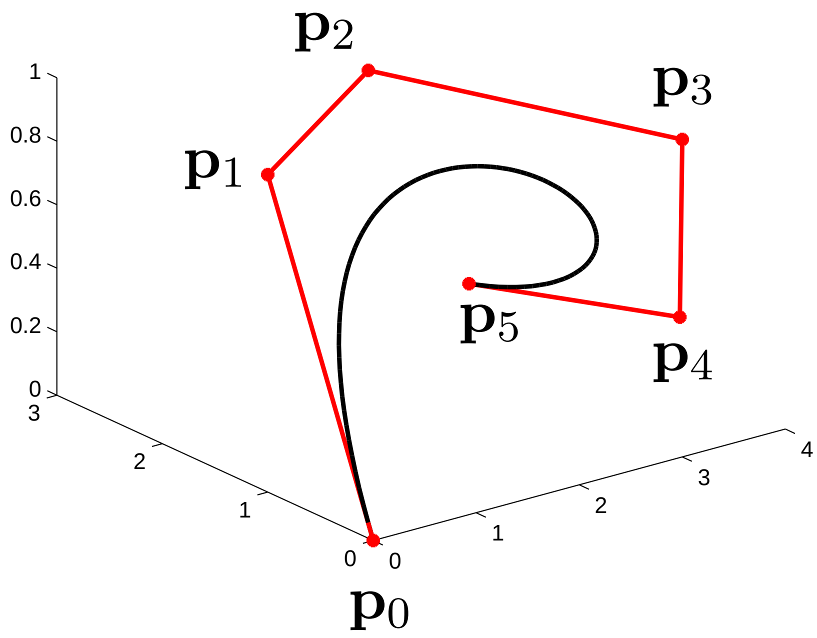


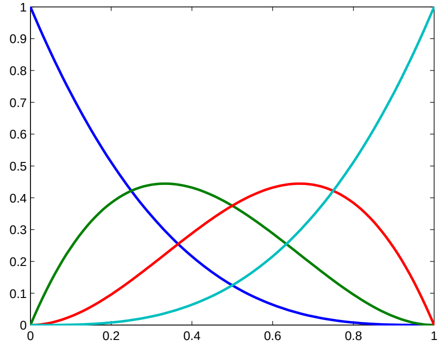
space: $< B_0, \cdots, B_n >$

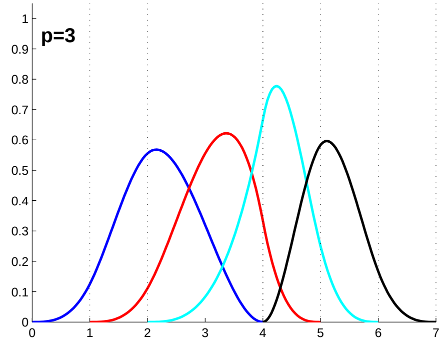
basis: ONTP basis (B-basis)

Optimal
Normalized
Totally
Positive

Bernstein-like representation

Bernstein/B-splines $\Rightarrow$ Optimal NTP bases





- Bernstein/B-splines bases are the ONTP bases for polynomials/piecewise polynomials

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  - optimal from a geometric point of view
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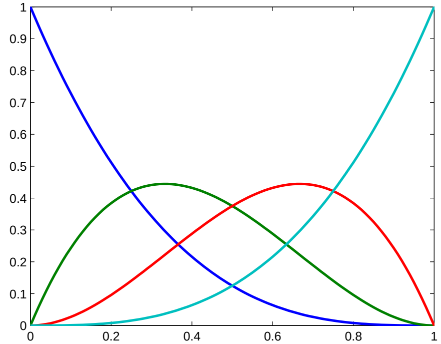
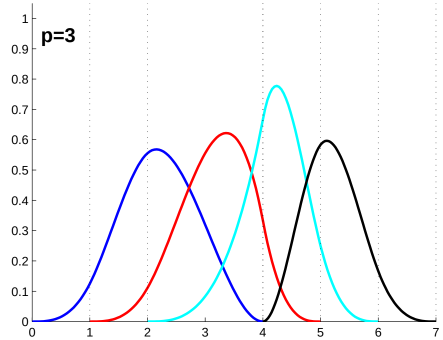
Bernstein/B-splines ⇒ Optimal NTP bases





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    exact reproduction of main curves/surfaces (conic sections, ...)
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Reproducing conic sections, cycloids ....

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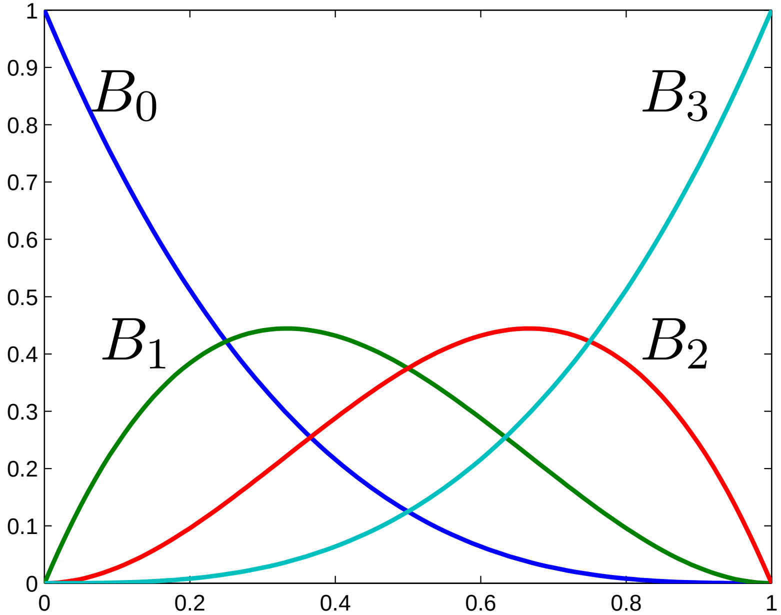

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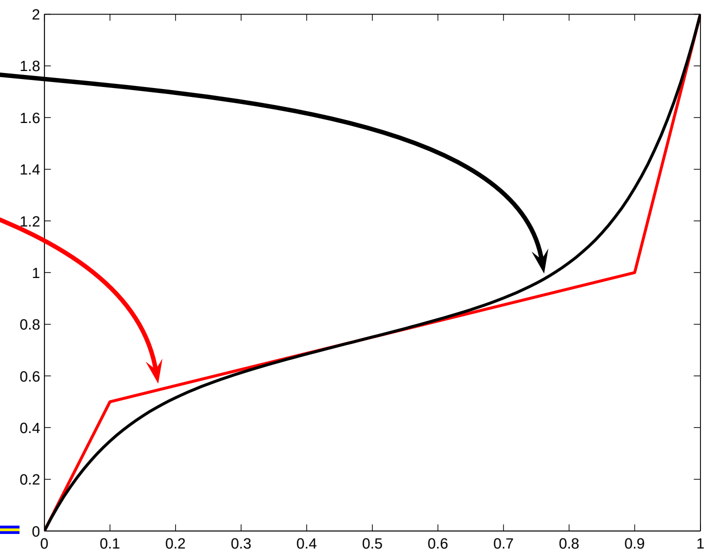

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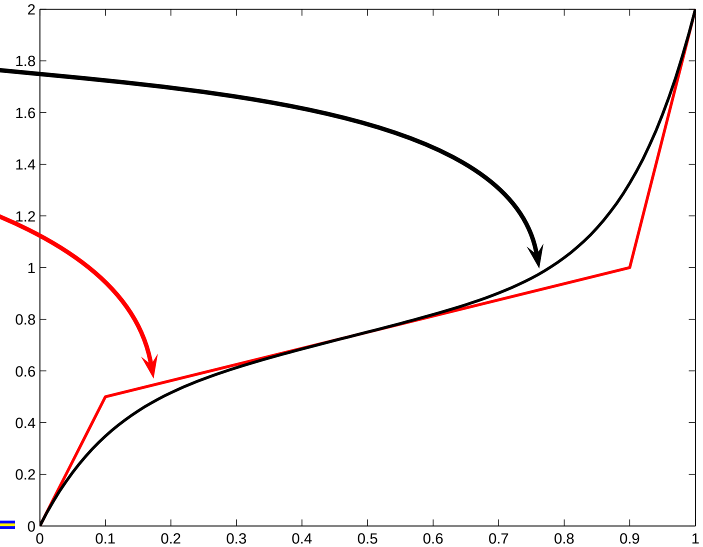
properties of s by its control polygon



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- Ex:
  - u, v: trigonometric functions
  - u, v: exponential functions
  - u, v: variable degree
  - ....

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Bernstein-like representations

[Goodman, T.N.T., Mazure, M.-L., JAT, 2001]

[Mainar, E., Peña, J.M., Sánchez-Reyes, J, CAGD 2001]

[Carnicer, Mainar, Peña; CA 2004]

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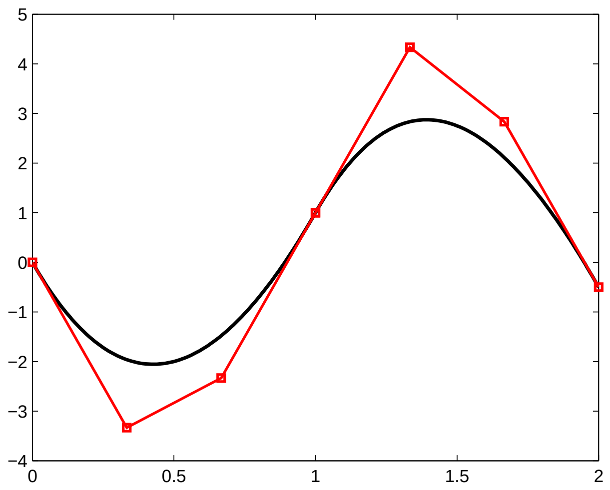
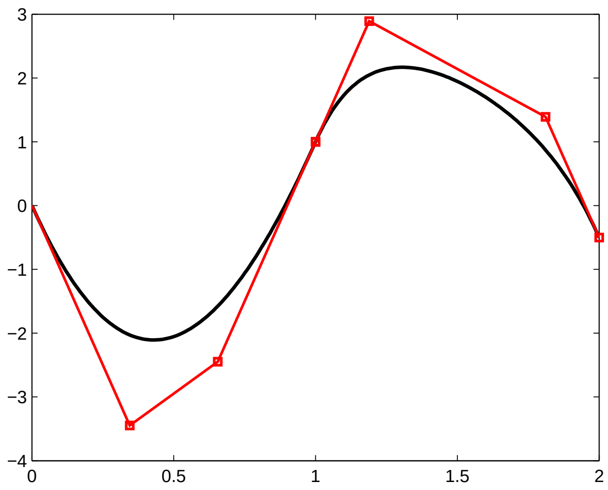


C^1 cubics



C^1 Trig/Exp

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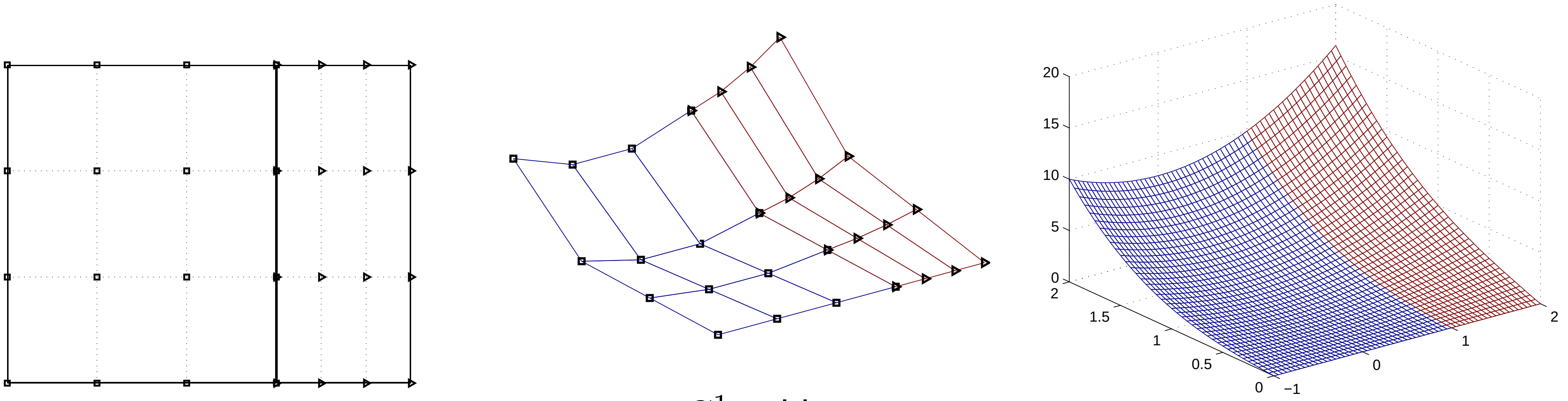

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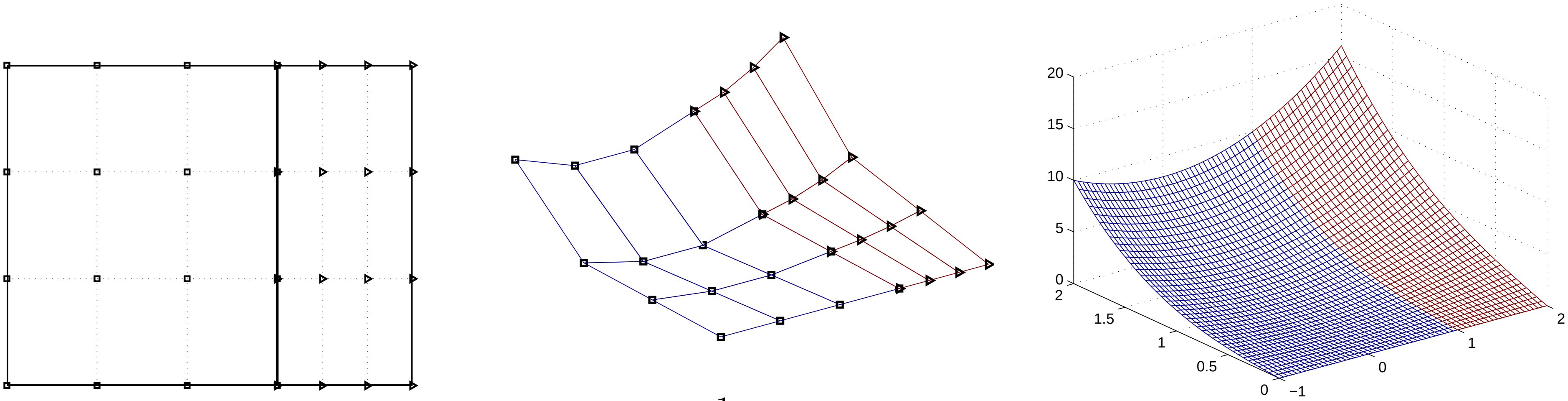

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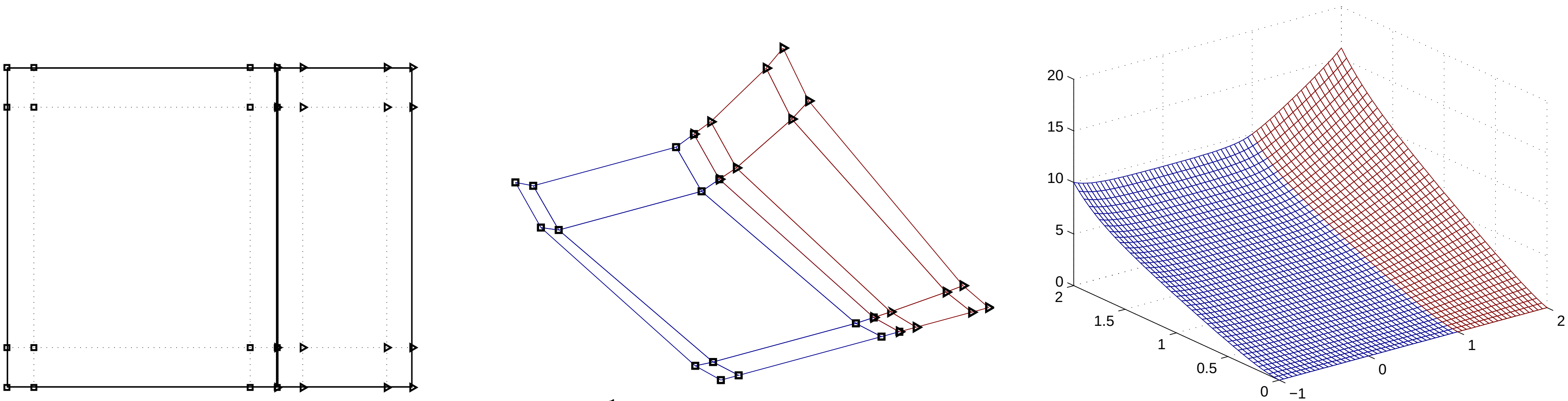

C^1 exponential (cubics)

Spaces good for design

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Spaces good for design

$\mathbb{E} \subset C^n$: $n+1$ dimensional EC space containing constants

$\mathbb{E}$ is Extended Chebyshev (EC) in I if any non trivial element has at most n zeros in I

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- in $\mathbb{E}$ all classical geometric design algorithms can be developed for the Bernstein-like basis (blossoms)
  $\Rightarrow$ $\mathbb{E}$ is good for design true under less restrictive hypoteses

[Goodman, T.N.T., Mazure, M.-L., JAT, 2001], [Carnicer, Mainar, Peña; CA 2004], [Mazure, M.-L., AiCM, 2004], [Mazure, M.-L., CA, 2005], [Costantini, P., Lyche, T., Manni, C., NM, 2005], [Mazure, M.-L., NM, 2011]

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$$\mathbb{P}_p^{u,v} \quad := \quad < 1, t, \ldots, t^{p-2}, \cosh \omega t, \sinh \omega t >= \text{HYP}$$

conic sections, helix, cycloid, ...

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- rational model: $\mathbb{P}_p \rightarrow$ B-splines $\rightarrow$ NURBS
- alternative: $\mathbb{P}_p = < 1, t, \ldots, t^{p-2}, t^{p-1}, t^p >$

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$$\mathbb{P}_p^{u,v} := < 1, t, \ldots, t^{p-2}, u(t), v(t) >$$

- select proper $\mathbb{P}_p^{u,v}$:
  - good approximation properties
  - describe sharp variations

$$\mathbb{P}_p^{u,v} \quad := \quad < 1, t, \ldots, t^{p-2}, e^{\omega t}, e^{-\omega t} >= \text{EXP} = (\text{HYP})$$

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- construct/analyse spline spaces with sections in $\mathbb{P}_p^{u,v}$ with suitable bases for them (analogous to B-splines)

[Lyche, CA 1985]

[Schumaker, L.L.; 1993],

[Koch, P.E, Lyche, T.; Computing 1993],

[Marušic, M., Rogina, M.; JCAM 1995],

[Kvasov, B.I., Sattayatham, P.; JCAM 1999],

[Costantini, P.; CAGD 2000],

[Costantini, P., Manni, C.; RM 2006]

[Wang Fang; JCAM 2008],

[Kavcic, Rogina, Bosner, Math. Comput. in Simulation, 2010], . . .

Generalized B-splines

$$\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\},$$

$\{..., \ u_i, v_i, \ ...\}, < 1, t, \ldots, t^{p-2}, u_i(t), v_i(t) >, < D^{p-1}u_i, D^{p-1}v_i >$ Chebyshev

$$D^{p-1}v_i(\xi_i) = 0, \quad D^{p-1}v_i(\xi_{i+1}) > 0, \qquad D^{p-1}u_i(\xi_i) > 0, \quad D^{p-1}u_i(\xi_{i+1}) = 0,$$

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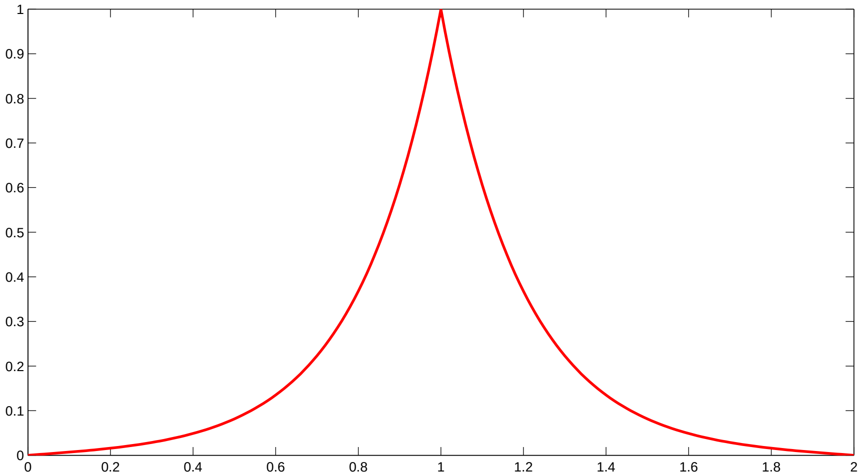
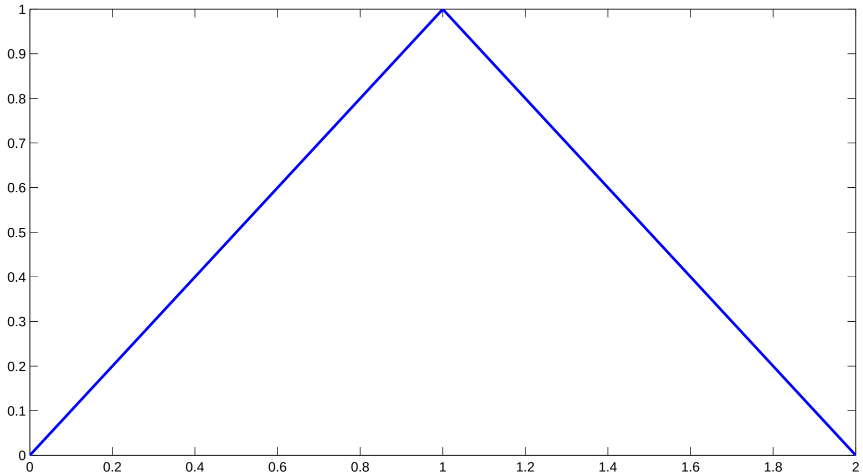


$\widehat{B}_{i,\Xi}^{(1)}$



$B_{i,\Xi}^{(1)}$

- All Chebyshevian spline spaces good for design can be built by means of integral recurrence relations, [Mazure M.L., NM 2011]

Generalized B-splines: exponential (hyperbolic)

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$$\mathbb{P}_p^{u_i,v_i} :=< 1, t, \ldots, t^{p-2}, \cosh \omega_i t, \sinh \omega_i t >$$

- Exponential case: $p = 3$

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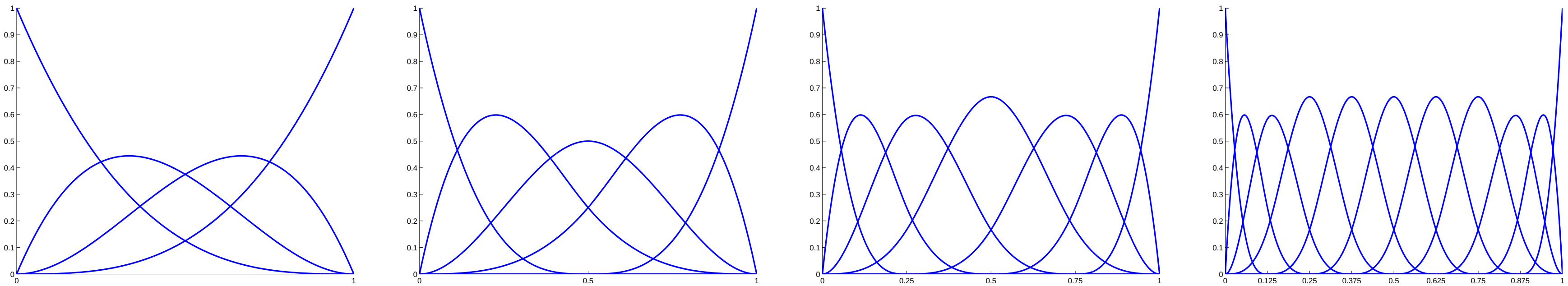
Bernstein-like basis



$\omega \to 0$: C^2 cubic B-splines

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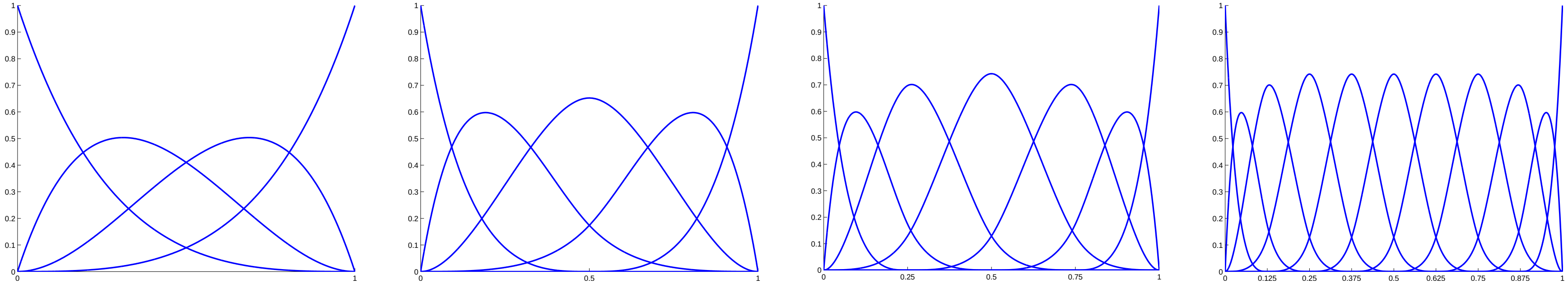

$\omega = 3h$

Generalized B-splines: properties

$$\{\widehat{B}_{i,\Xi}^{(p)}(t),\ i = 1, \ldots\},$$

- Properties analogous to classical B-splines
  - positivity
  - partition of unity: $p \geq 2$
  - compact support
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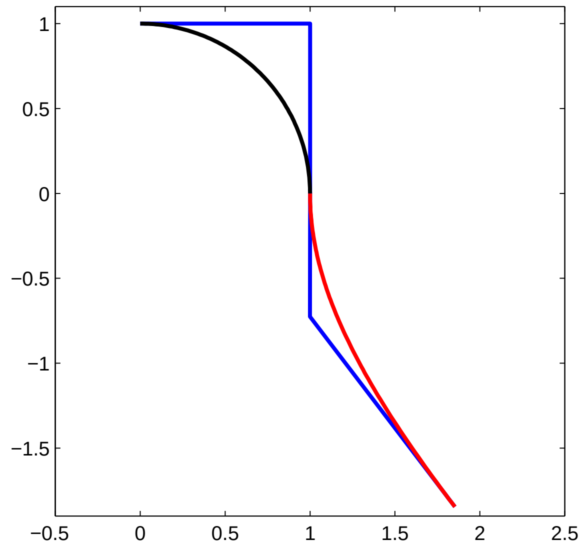

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  - straightforward multivariate extension via tensor product

Summary

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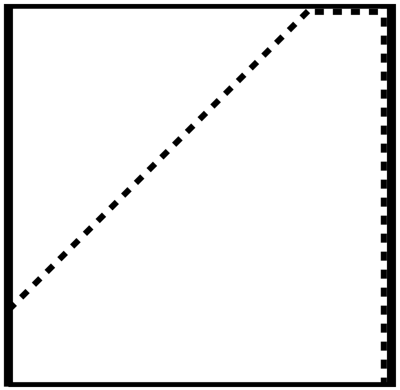

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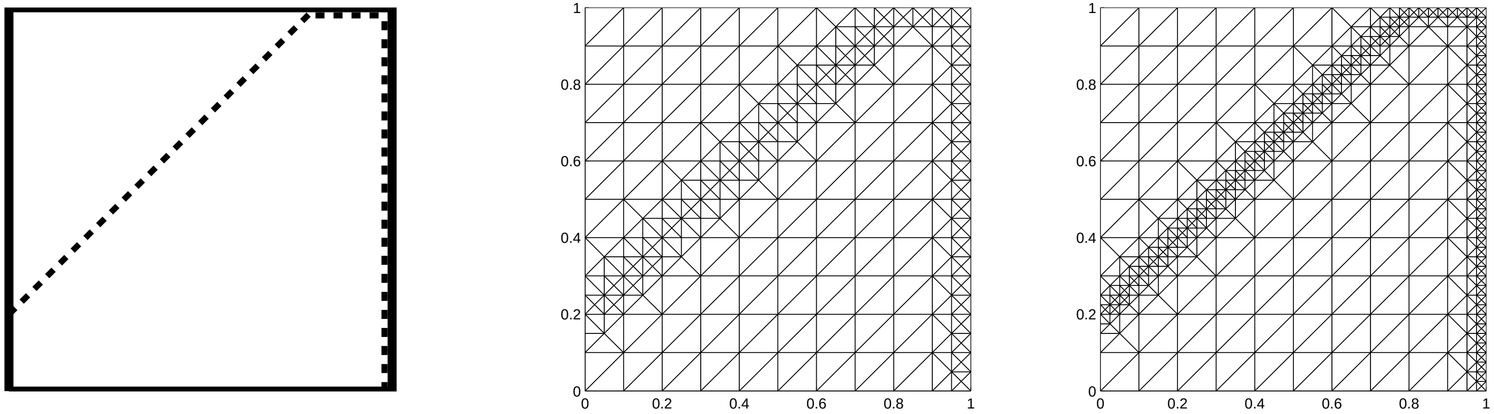

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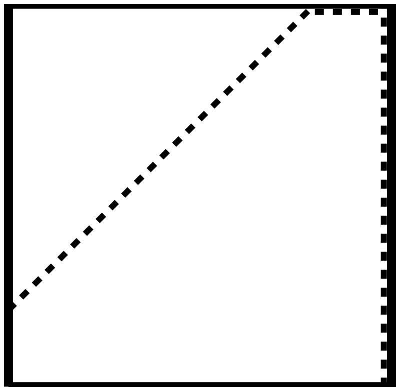

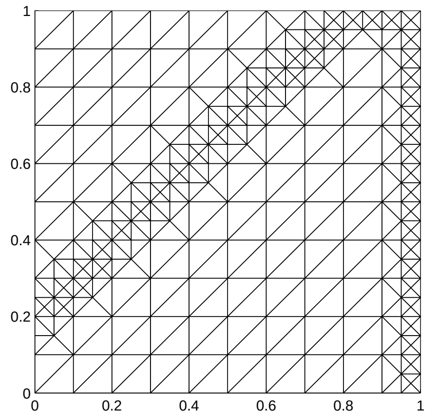

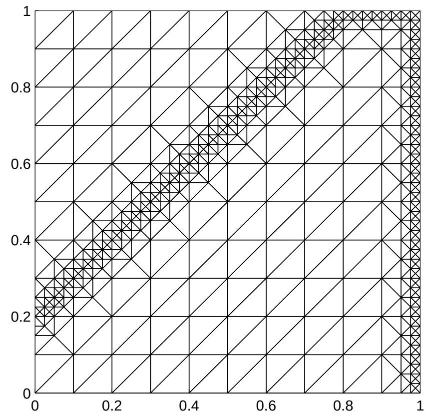

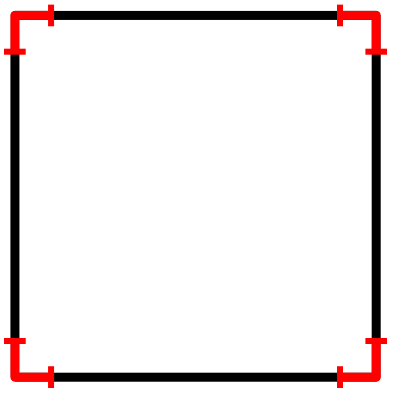

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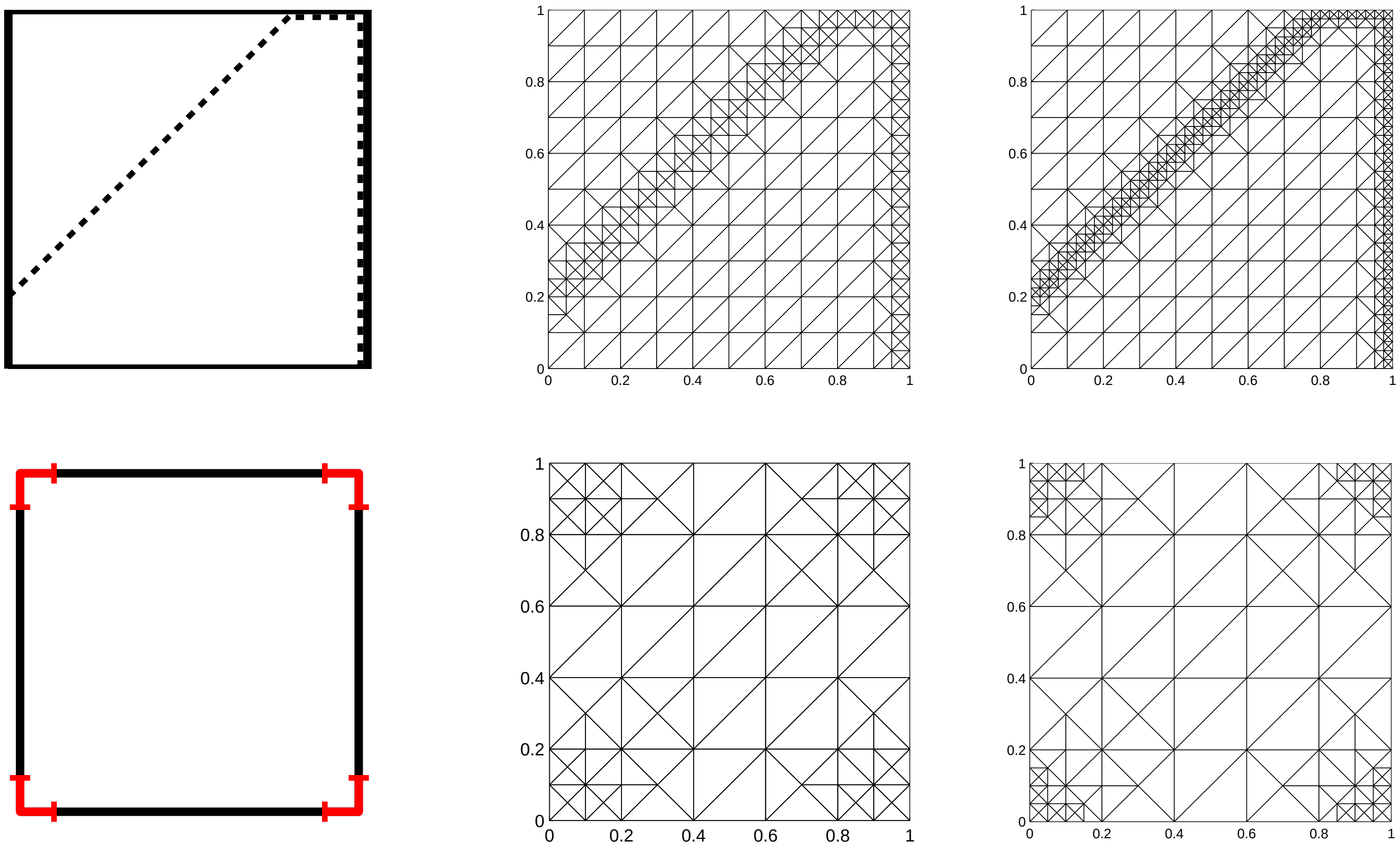

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- Generalized splines have global tensor-product structure
- some localization techniques can be applied to (some) generalized spline spaces.
  - Hierarchical generalized splines
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  - Quadratic Generalized splines over triangulations

Hierarchical model

[Forsey, D.R., Bartels R.H., CG 1988], [Kraft R., Bartels R.H., Surf. Fitt. Mult. Meth. 1997], [Rabut C., 2005]

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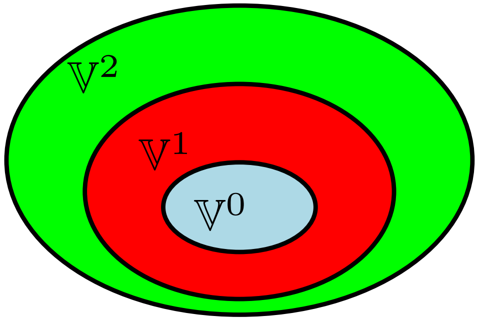
- sequence of N nested tensor-product spline spaces

$$\mathbb{V}^0 \subset \mathbb{V}^1 \subset \cdots \subset \mathbb{V}^{N-1}$$



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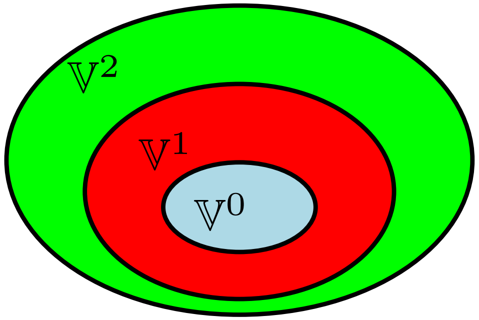

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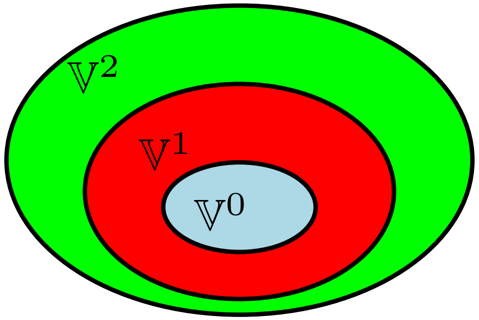

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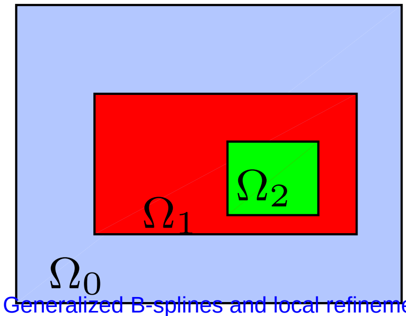
- sequence of N nested domains

$$\Omega_{N-1} \subset \Omega_{N-2} \subset \cdots \subset \Omega_0, \quad \Omega_N = \emptyset$$

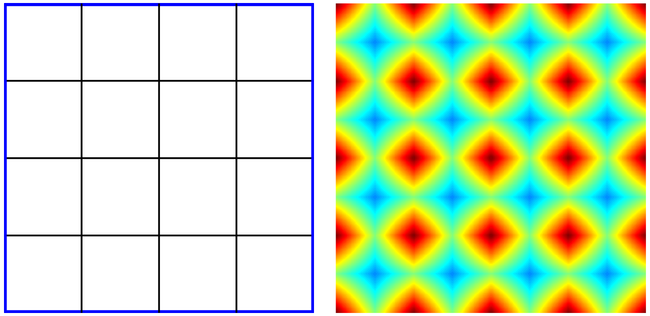


Hierarchical B-spline model

degree 1

Recursive definition

(I) Initialization: $\mathcal{H}^0 := \mathcal{B}^0$



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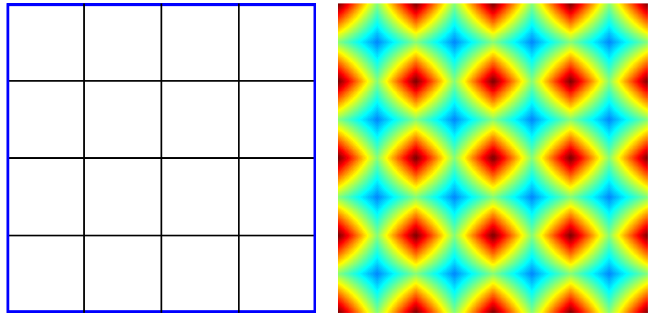

(II) construction of $\mathcal{H}^{\ell+1}$ from $\mathcal{H}^{\ell}$,

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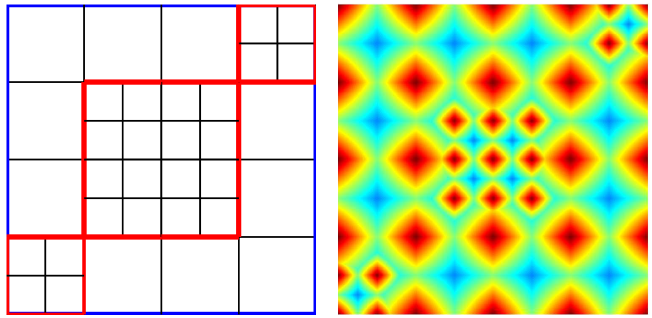

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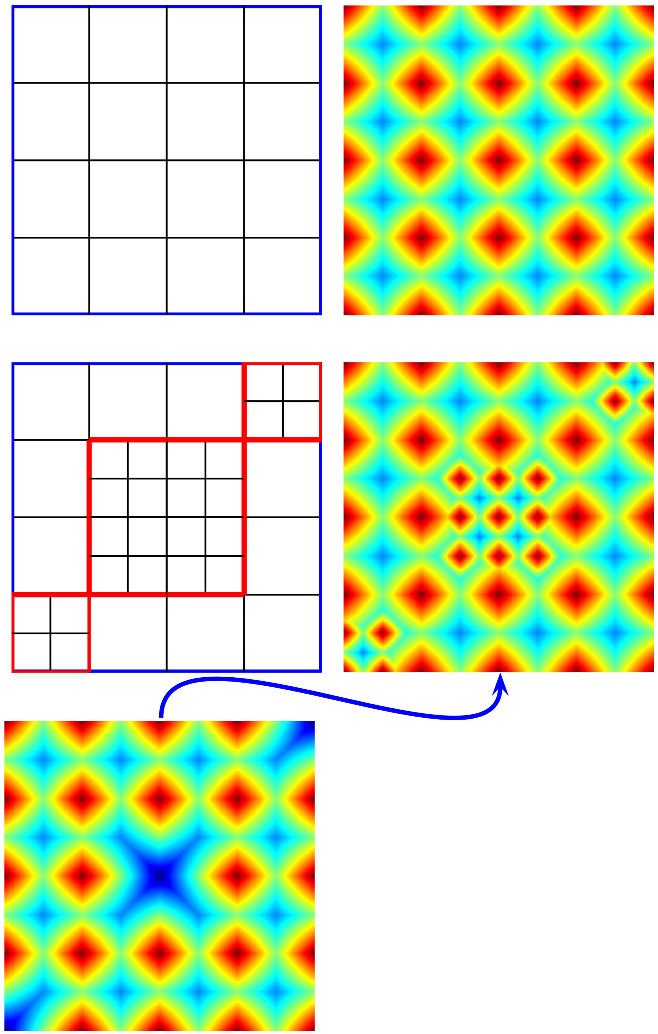

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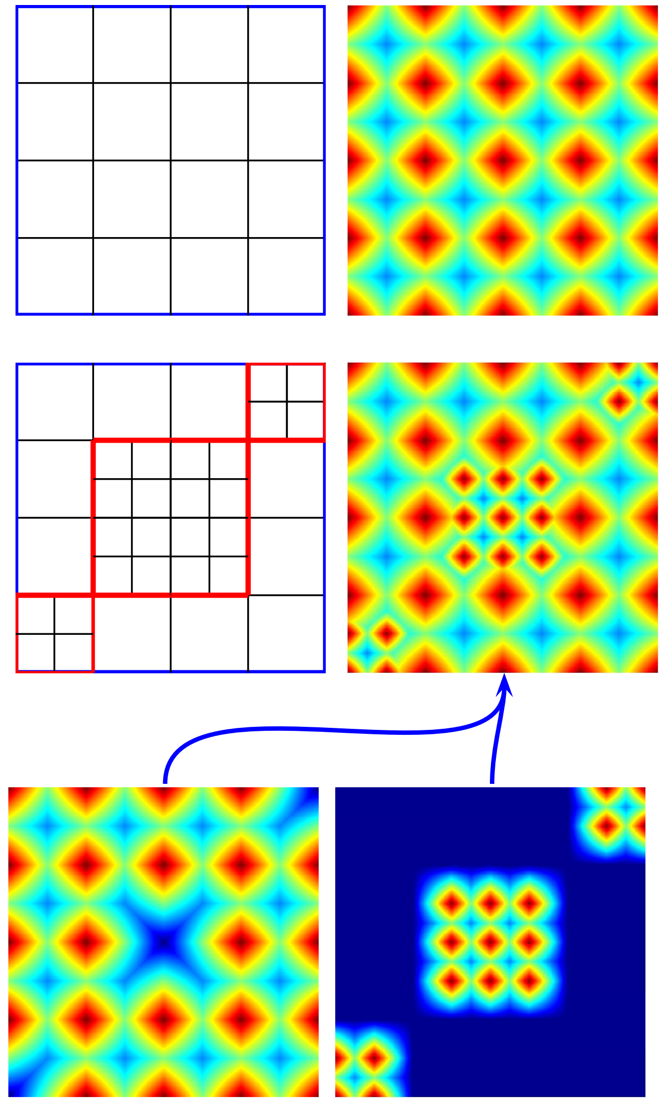

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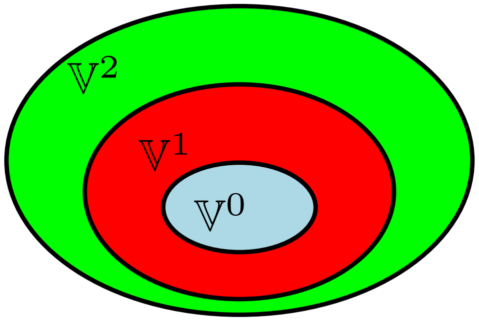

$\mathbb{V}^\ell$ is spanned by a tensor-product B-spline basis $\mathcal{B}^\ell$:

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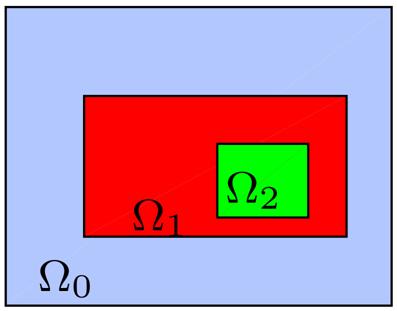

Hierarchical Generalized B-spline model

Generalized B-splines support a hierarchical refinement

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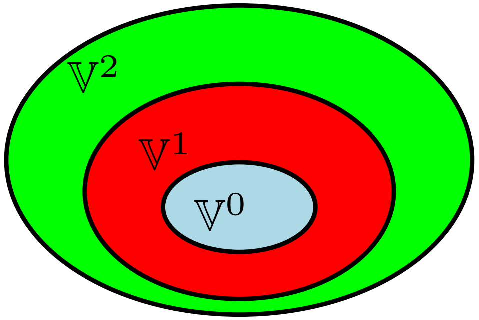

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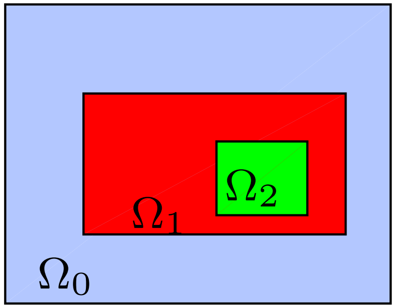

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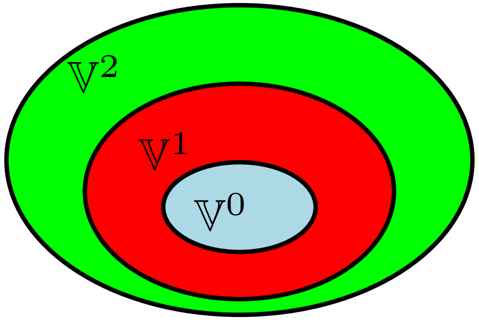

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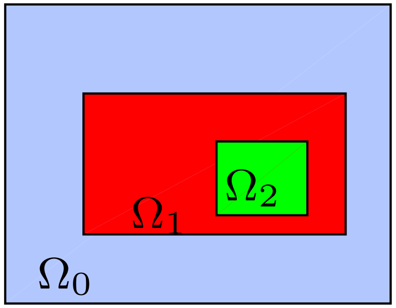

$\Rightarrow$ similar recursive definition

Hierarchical B-splines model

1D Example: Cubic B-spline basis

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]

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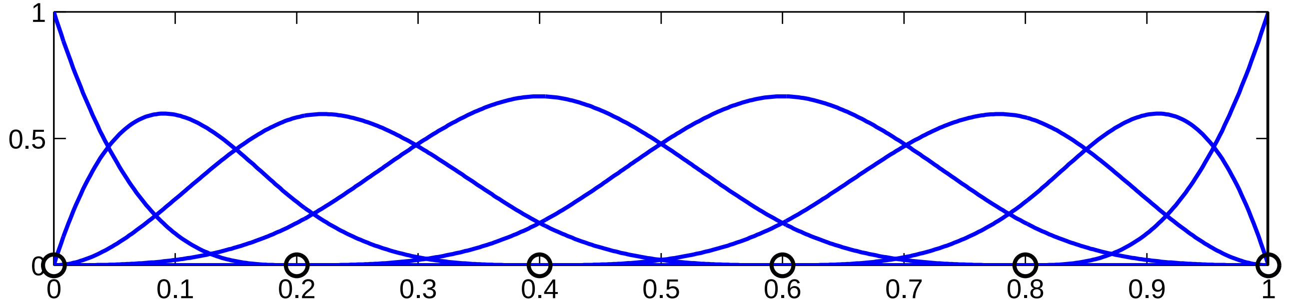

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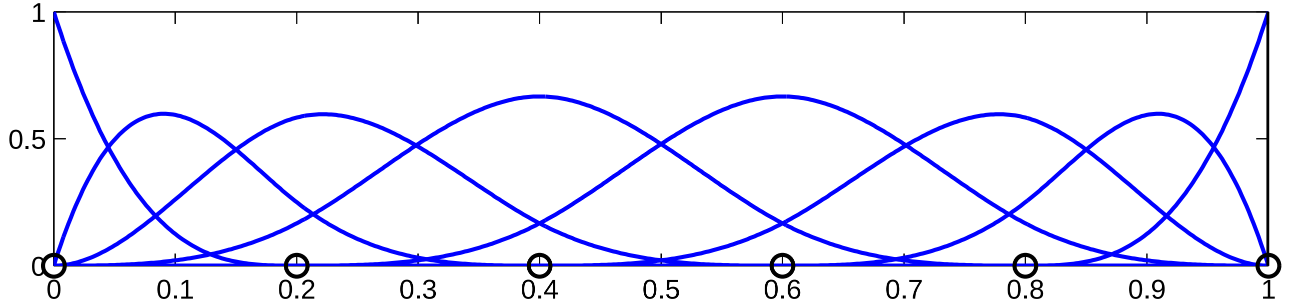

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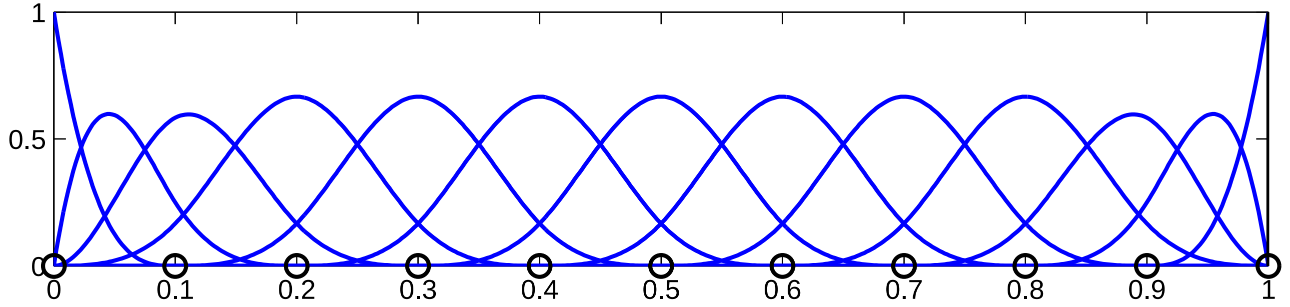

$\mathcal{B}^1$

[Vuong A.-V., Giannelli C., Jüttler B., Simeon B.; CMAME 2011]

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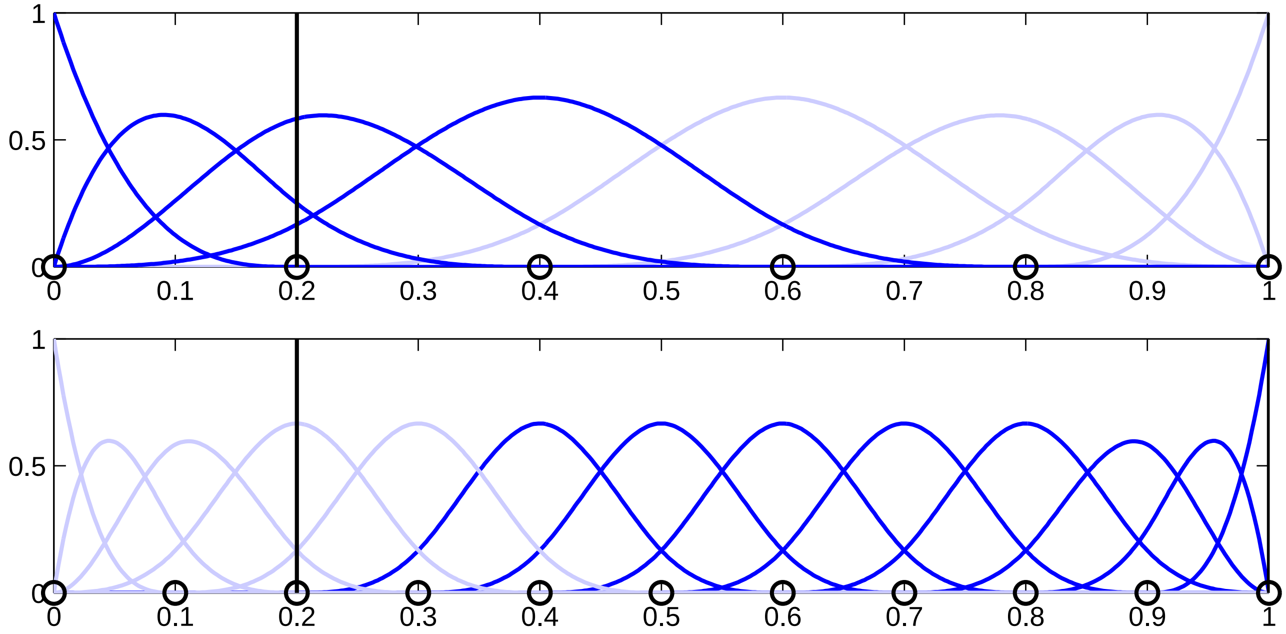

$\mathcal{H}_C^1$

$\cup$

$\mathcal{H}_F^1$

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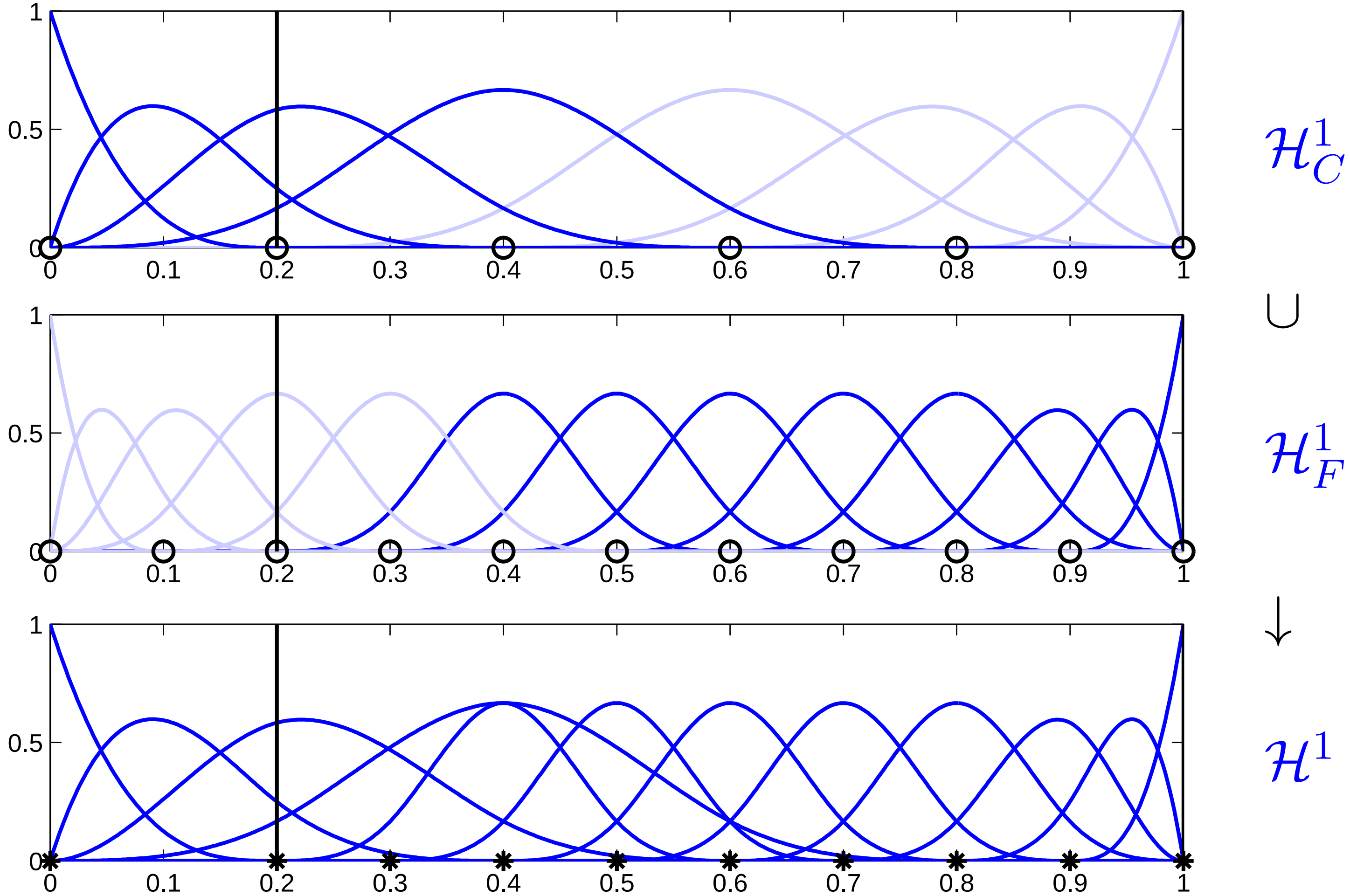

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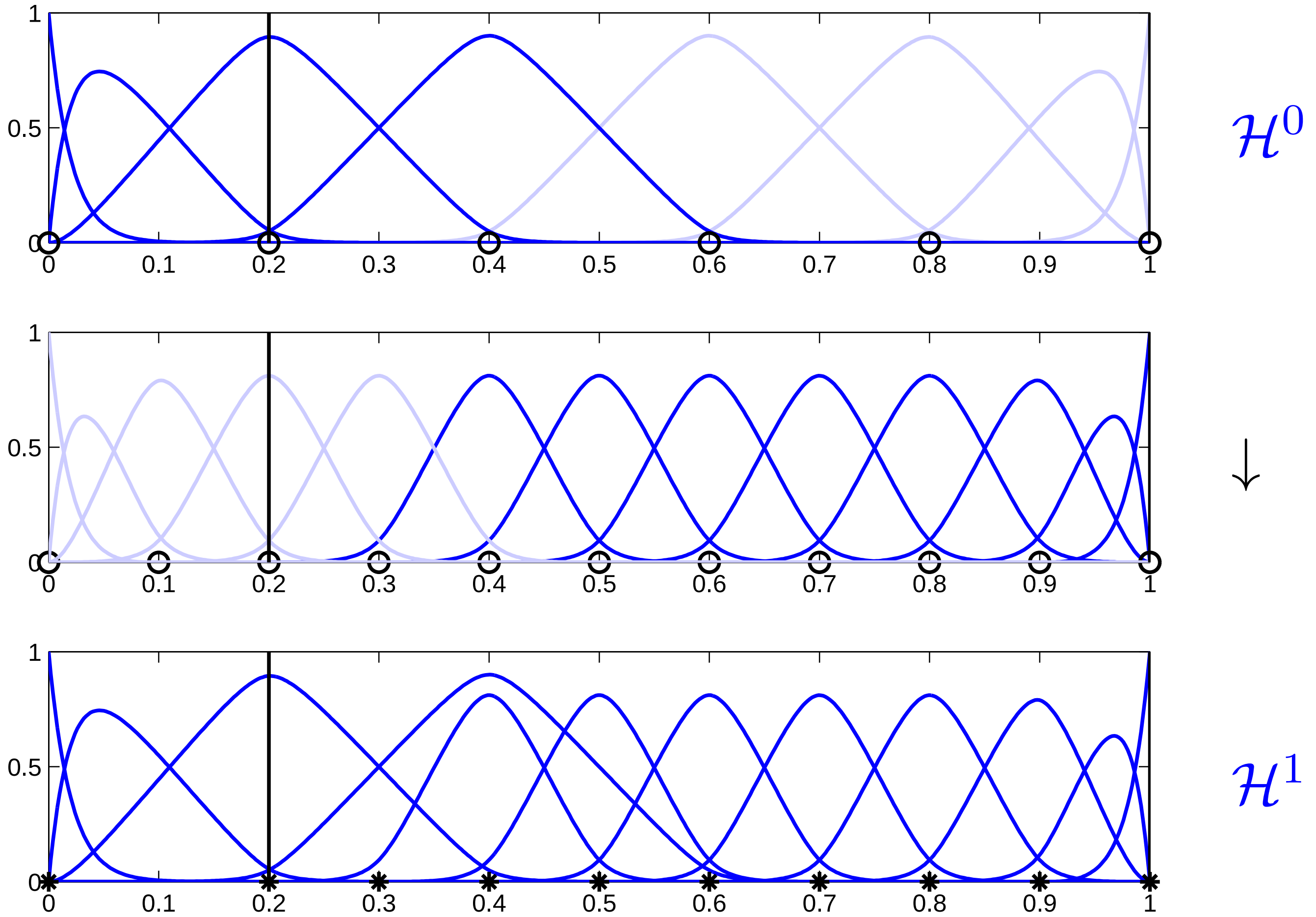

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The main properties of Hierarchical B-splines are inherited by Hierarchical GB-splines

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  - by using truncated bases

    [Giannelli, Jüttler, Speleers; AiCM 2013]

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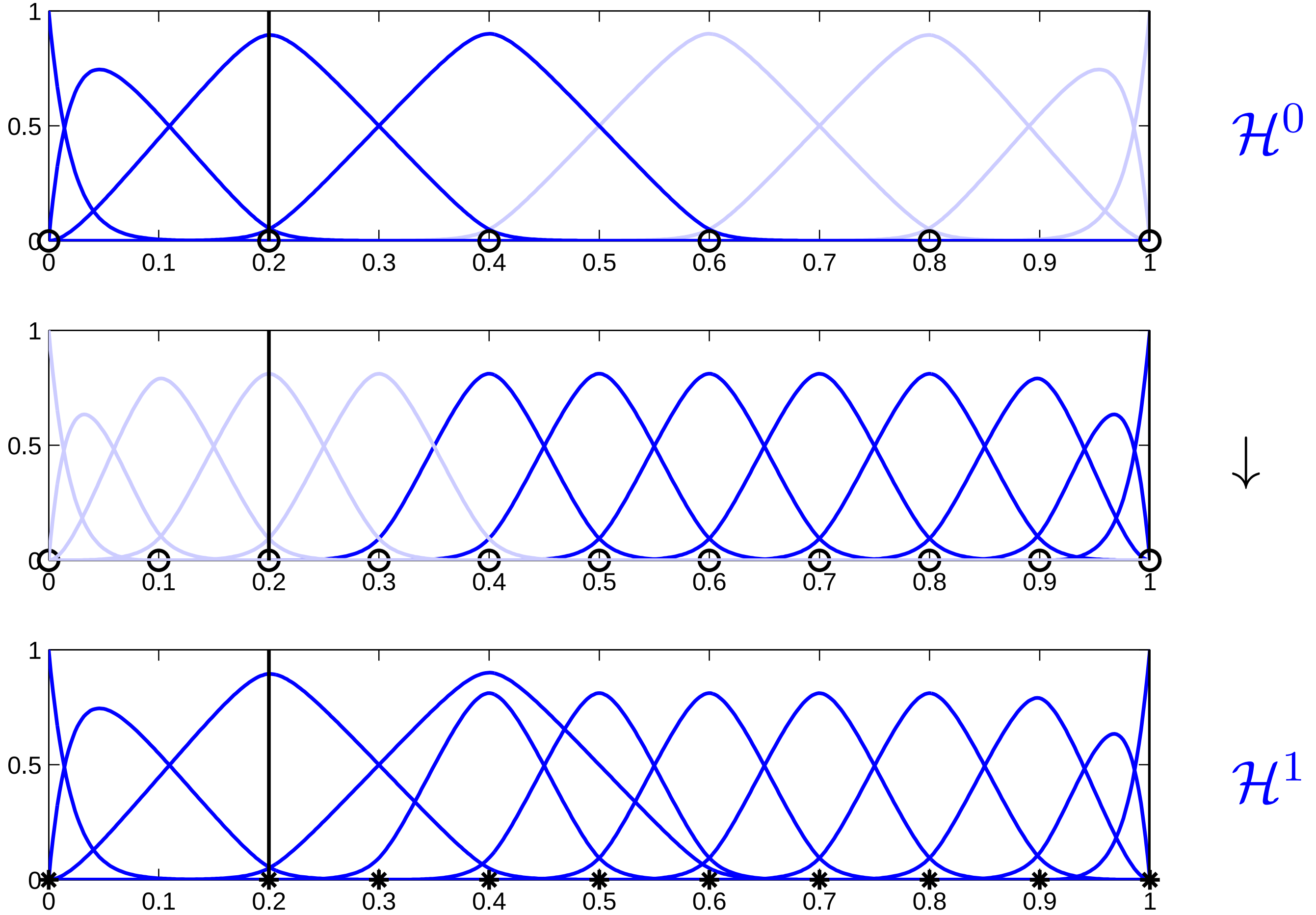

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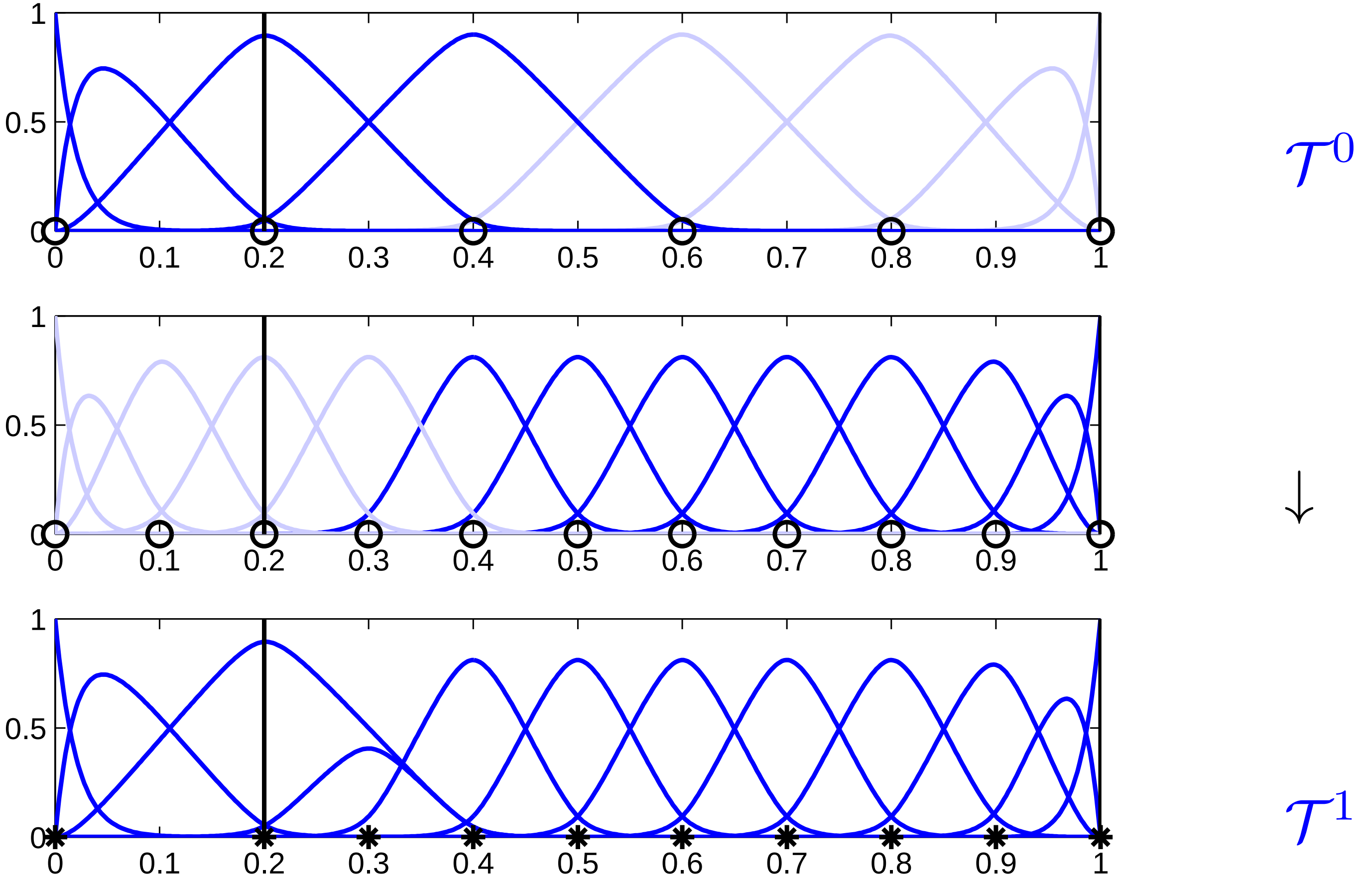

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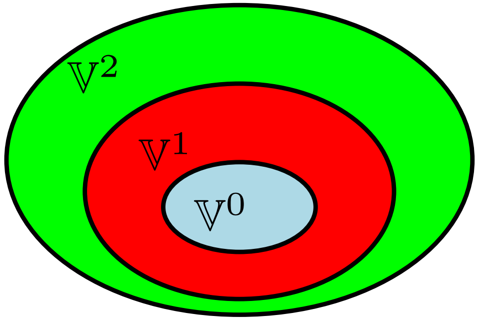

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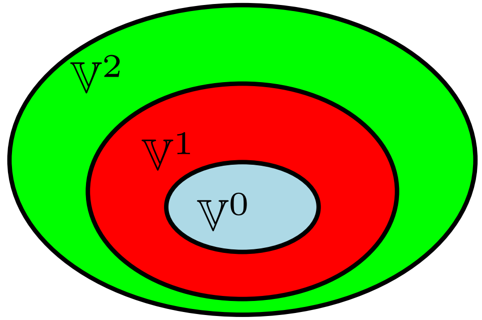

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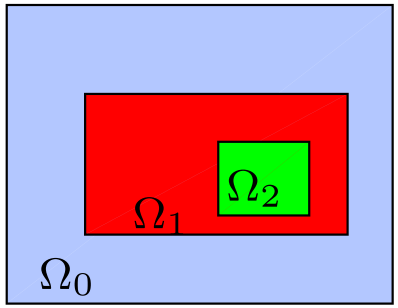

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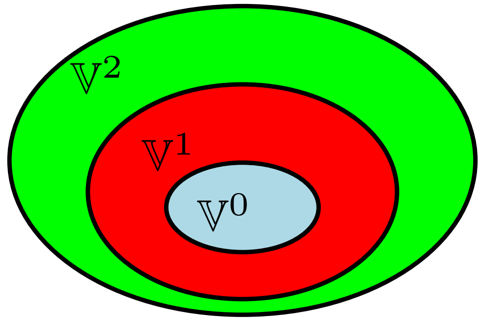

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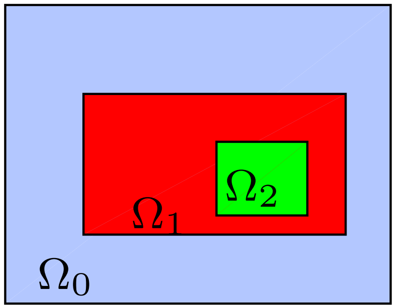

hierarchical (Generalized) B-splines span the full space

$$\{f : f|_{\Omega_0 \setminus \Omega_{\ell+1}} \in \mathbb{V}^\ell|_{\Omega_0 \setminus \Omega_{\ell+1}}, \ell = 0, \cdots, N-1\}$$

[Giannelli, Jüttler; JCAM 2013], [Speleers, Manni, 2013 preprint]

Hierarchical structures: not nested spaces

the construction can be applied to a hierarchy of not nested spaces

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  [Manni, Pelosi, Speleers; 2013, to appear]

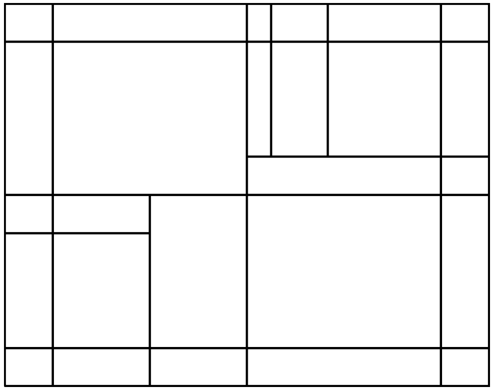
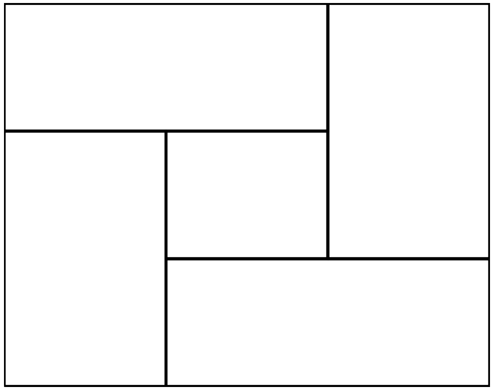
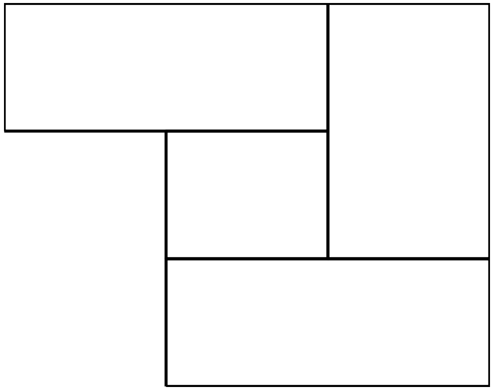
- Hierarchical B-splines are particular bases of particular spline spaces on special rectangular partitions

Spline spaces over T-meshes

Spline spaces over T-meshes

- T-mesh $\mathcal{T}$

  partition of a (rectangular) domain by rectangles: T-junctions (hanging vertices) are allowed

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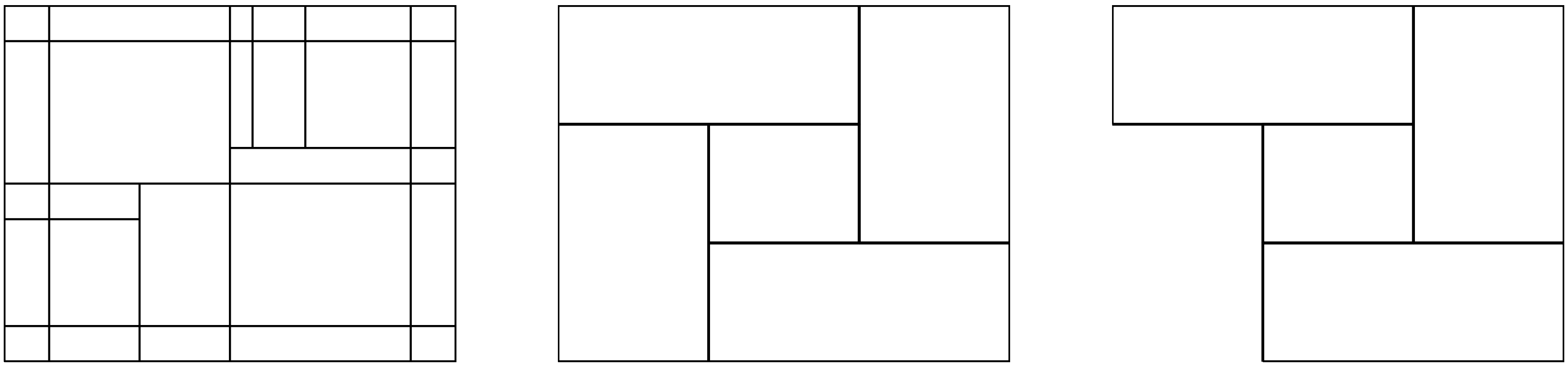

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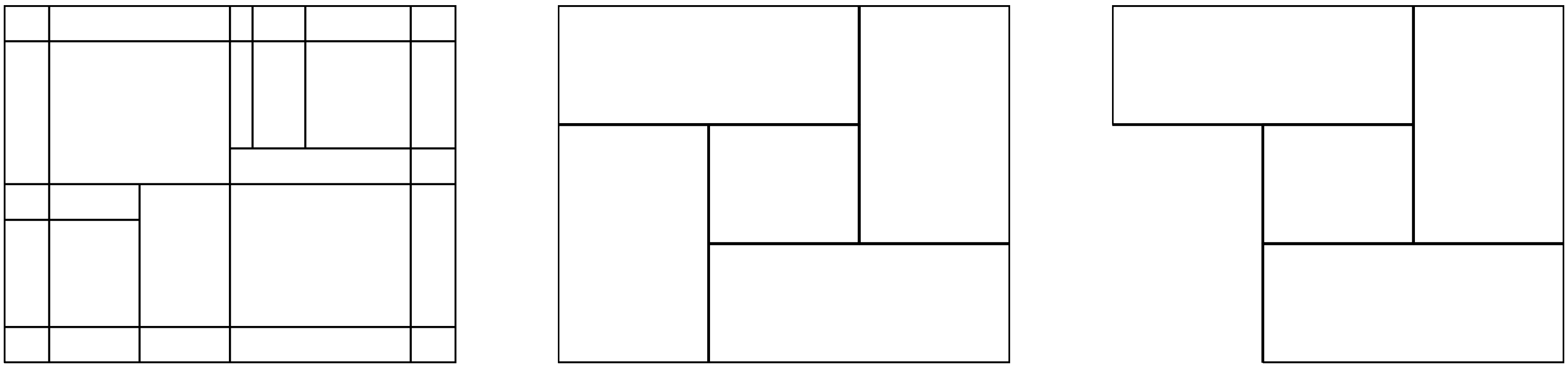

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✓ polynomial reproduction

∼ dimension?

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Spline spaces over T-meshes: dimension

- [Mourrain, B., Math. Comp. 2013]

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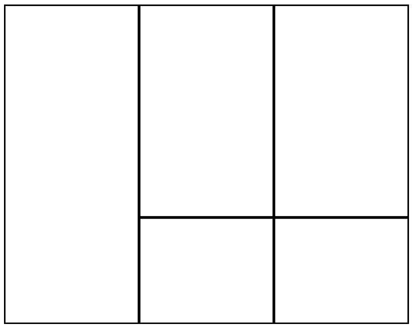
- C^1 cubics: $\dim(\mathbb{S}_3^1(\mathcal{T})) = 4(V_b + V_+)$

  $V_b : \#b.\ vertices, V_+ : \#cross.\ vertices$

  Ex: $\dim(\mathbb{S}_3^1(\mathcal{T})) = 4(9+1)$



Splines over T-meshes: dimension

- $\mathbf{d} \geq 2\mathbf{r} + 1$, rectangular domains: results based on
  - Bernstein representation
  - minimal determining sets

[Alfeld, P., Schumaker, L.L., CA 1987]

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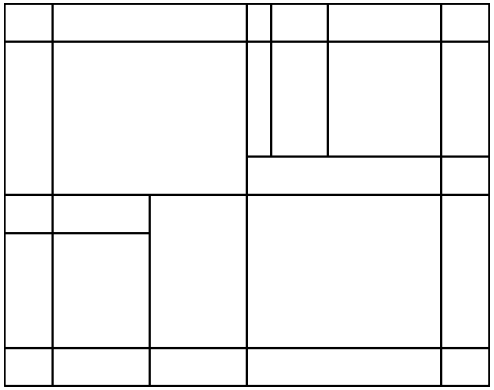

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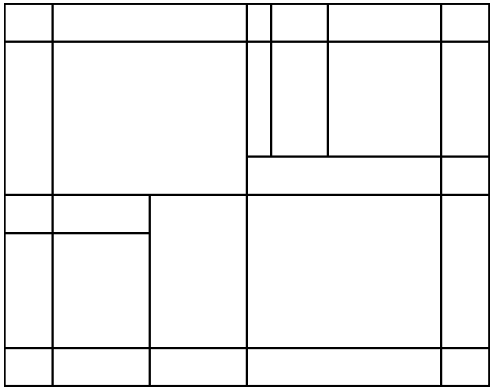

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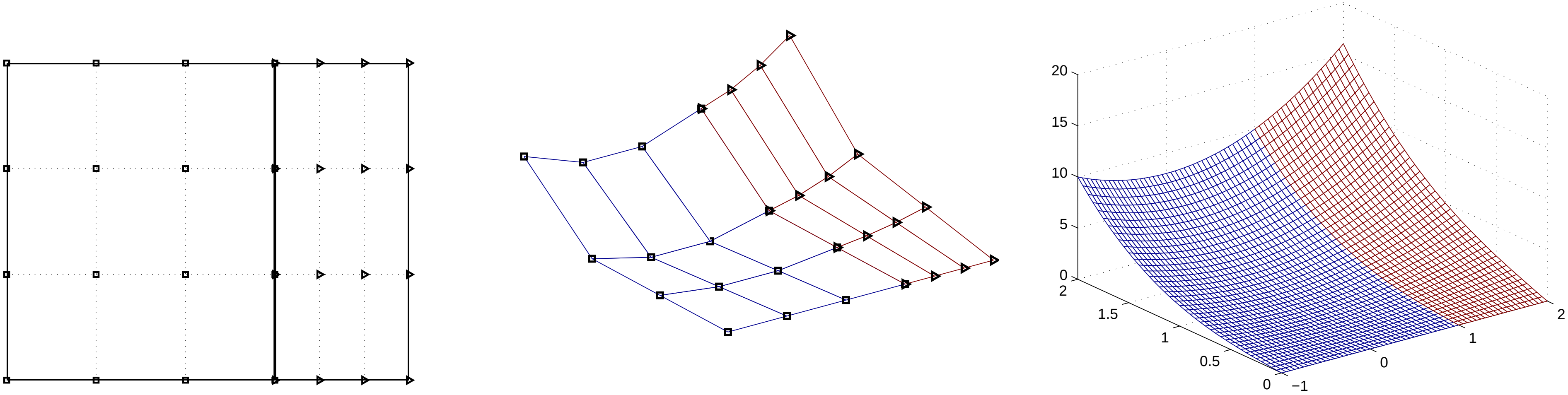

C^1 cubics

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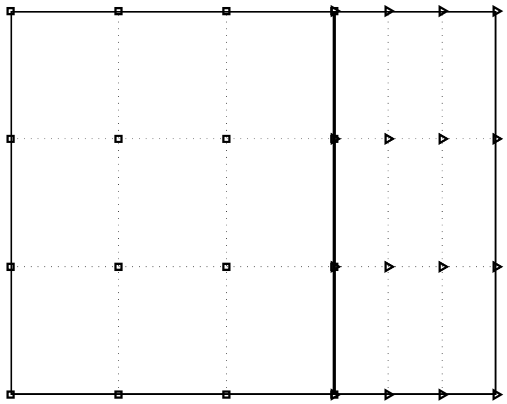

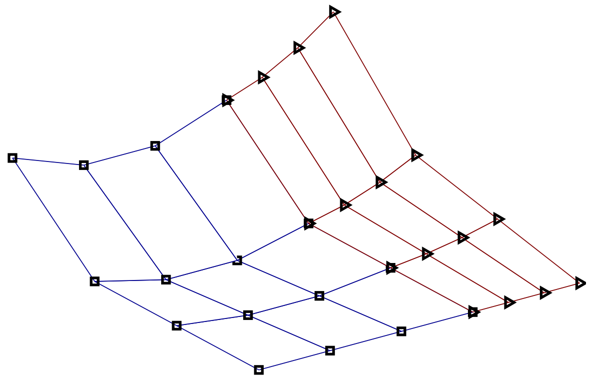

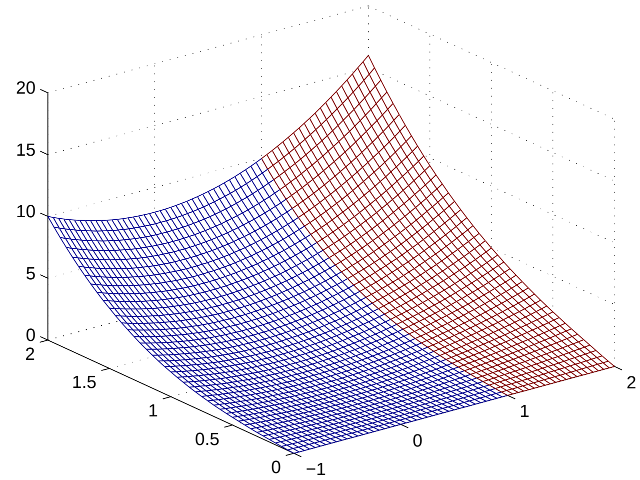

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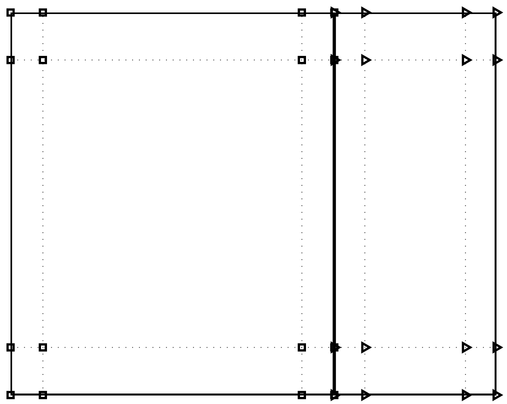
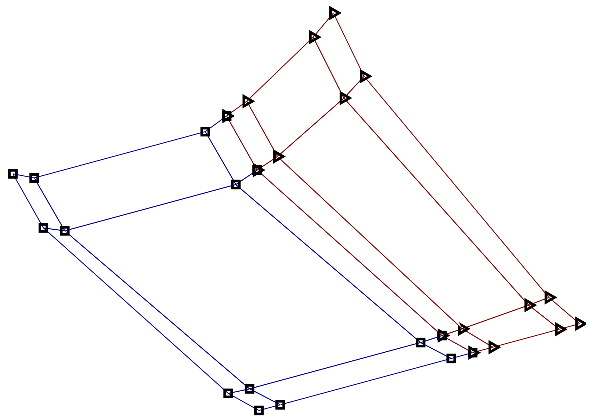
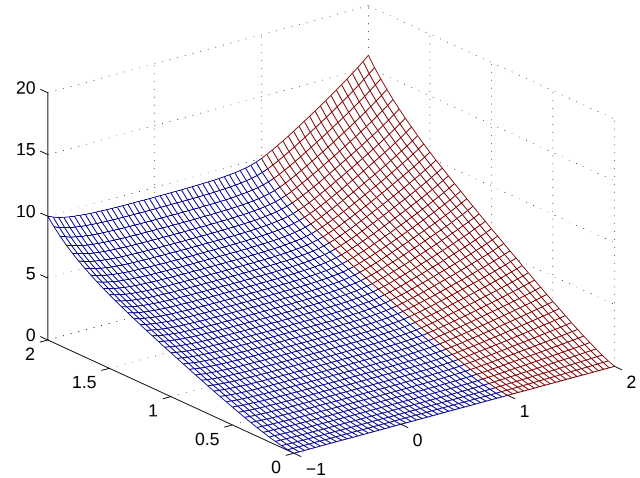
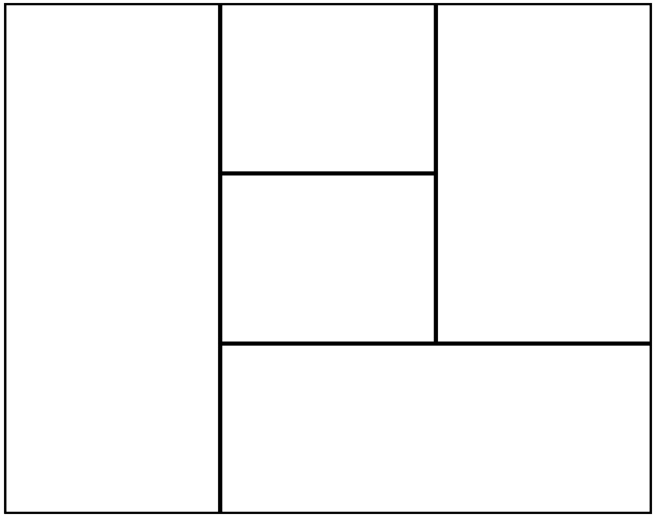




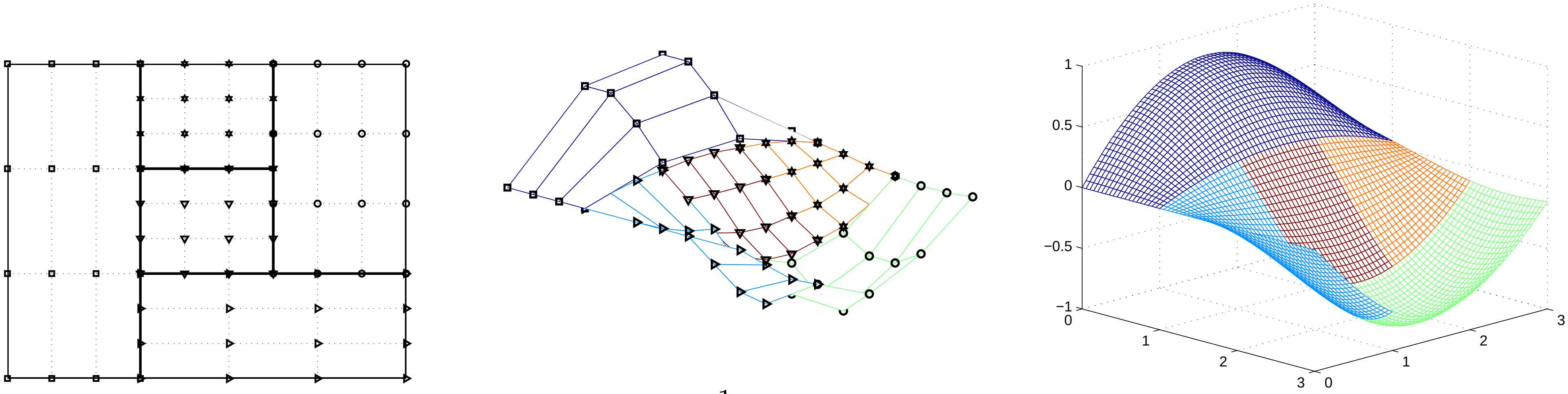


C^1 exponential (cubics)

Generalized Splines over T-meshes

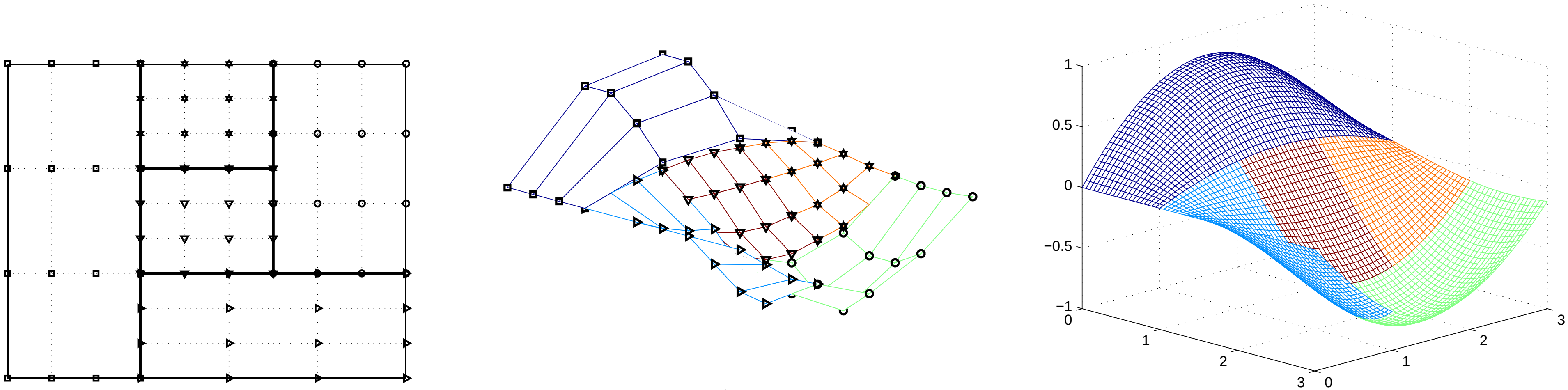


Generalized Splines over T-meshes

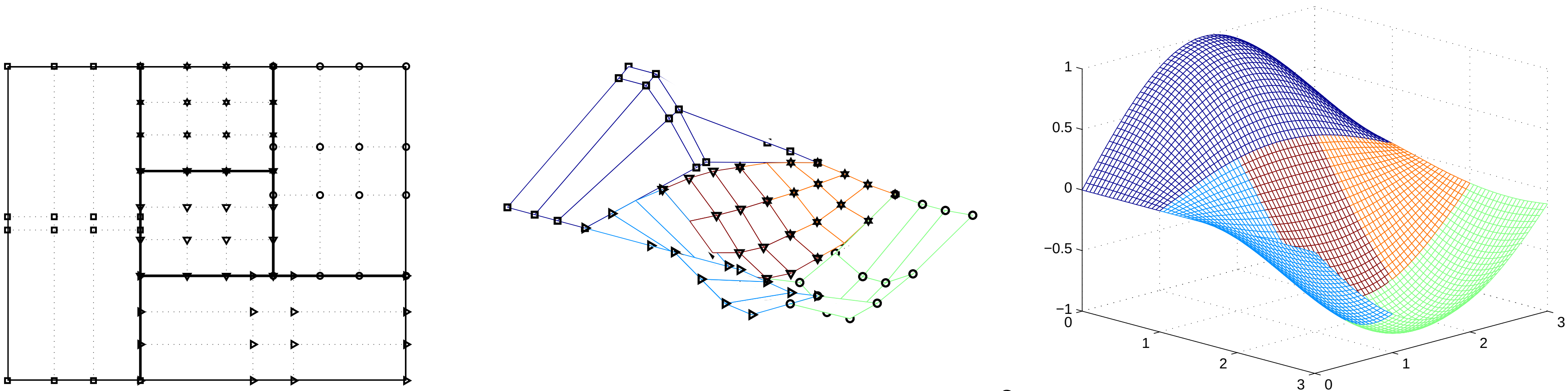


C^1 cubics

Generalized Splines over T-meshes



C^1 cubics



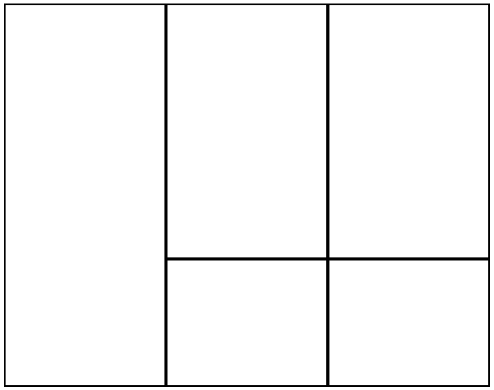
C^1 trigonometric (cubics), $\omega = \frac{2}{5}\pi$

Generalized Splines over T-meshes: dimension

- trigonometric/exponential C^1 cubics:

$$\dim(\widehat{\mathbb{S}}_3^1(\mathcal{T})) = 4(V_b + V_+)$$

$$V_b : \#b.\ vertices, V_+ : \#cross.\ vertices$$



$$\dim(\mathbb{S}_3^1(\mathcal{T})) = 4(9 + 1)$$

So far so good...

- Hierarchical bases, T-meshes: similar behavior of B-splines/GB-splines

- Hierarchical bases, T-meshes: similar behavior of B-splines/GB-splines
- Triangulations?

Quadratic Generalized Splines over Triangles

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- Bernstein like representation
  control polygon for functions?

$$t \notin < 1, u(t), v(t) >$$

No Greville abscissae

Quadratic Generalized Splines over Triangles

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- control points $f = b_0 B_0 + b_1 B_1 + b_2 B_2 \in \mathbb{P}_2^{u,v}$

$$\Downarrow$$

$$(0, b_0),\ (\xi, b_1), (1 - \xi, b_1), (1, b_2) \quad B_0(t) = B_2(1 - t) \ \ \xi = -1/B_0'(0) = 1/B_2'(1)$$

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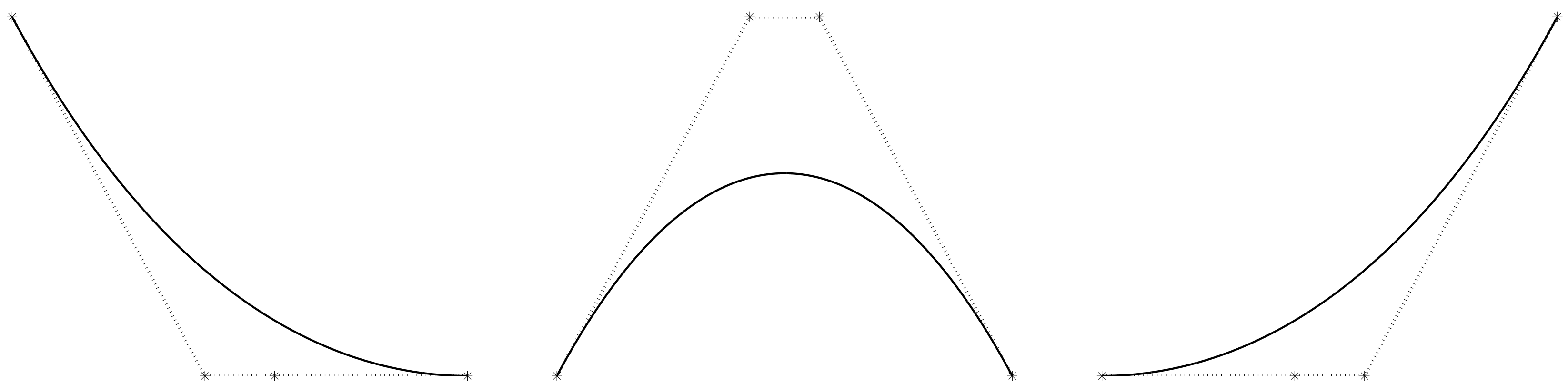

Quadratic Generalized Splines over Triangles

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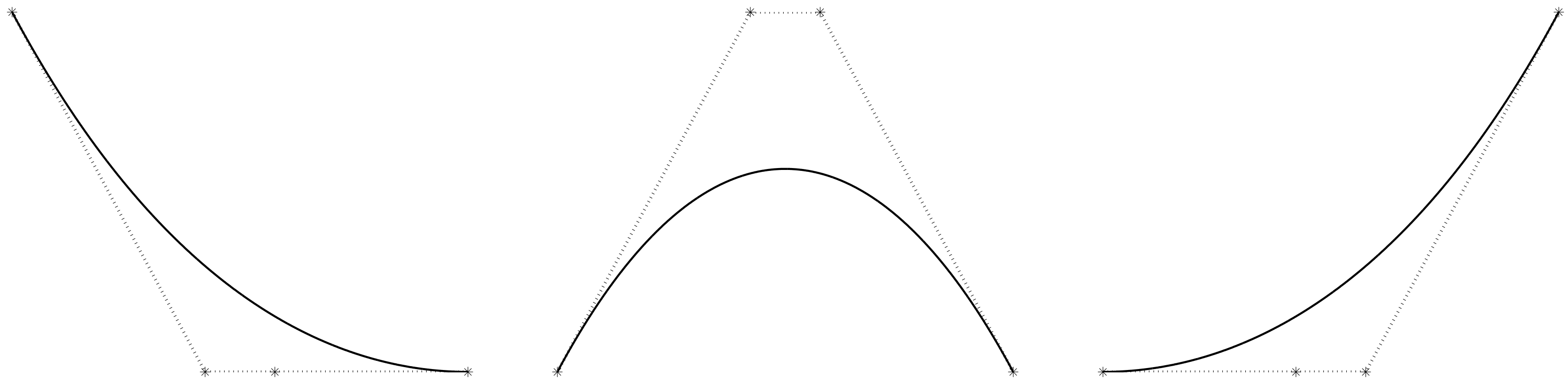

- geometric properties of the usual control polygon

Quadratic Generalized Splines over Triangles

$$\mathbb{H}_\omega := < 1, \cosh \omega t, \sinh \omega t >, \; t \in [0,1]$$

Quadratic Generalized Splines over Triangles

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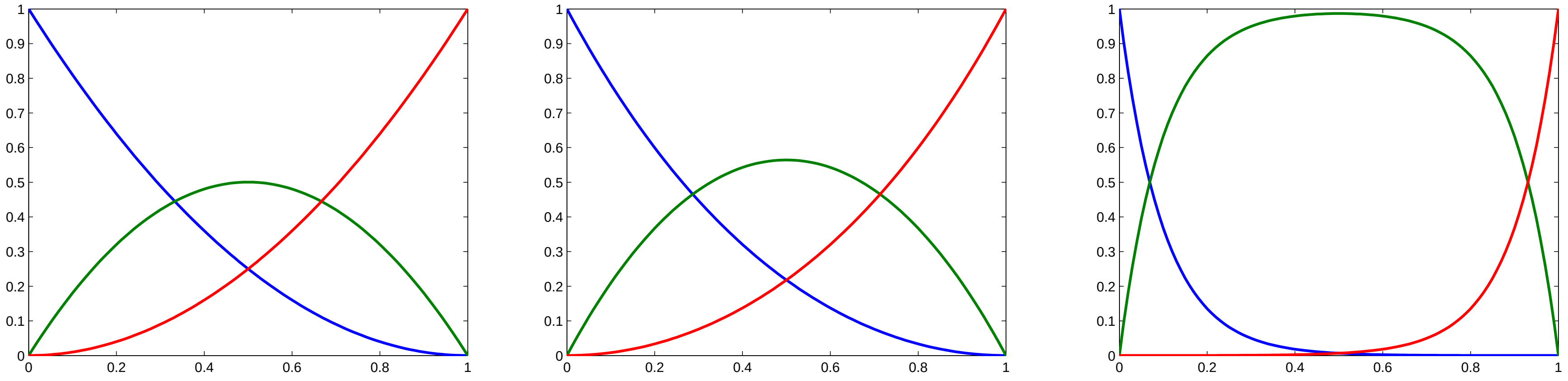

$\omega = 0.1$ $\omega = 1.5$ $\omega = 10$

Quadratic Generalized Splines over Triangles

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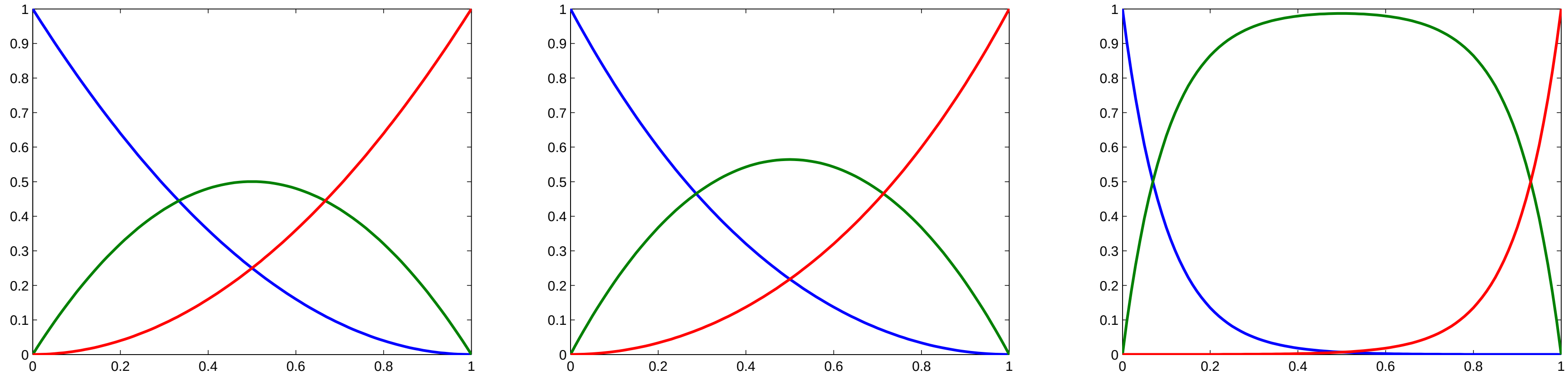

$\omega = 0.1$ $\omega = 1.5$ $\omega = 10$

ONTP basis $B_{0,\omega}, \ B_{1,\omega}, \ B_{2,\omega}, \omega \to 0$ quadratic Bernstein pol.

Quadratic Generalized Splines over Triangles

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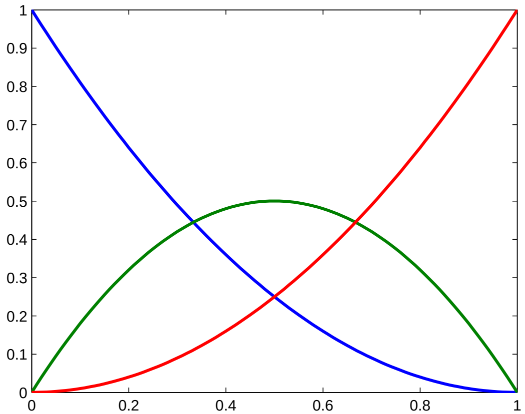
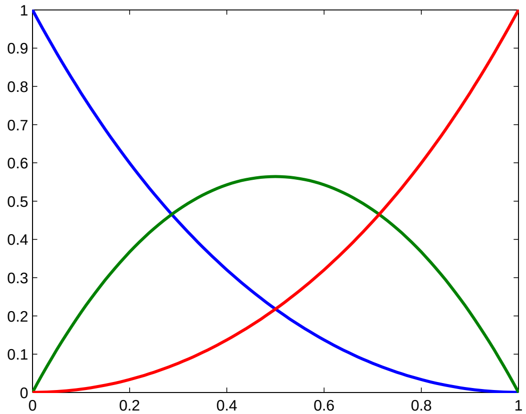
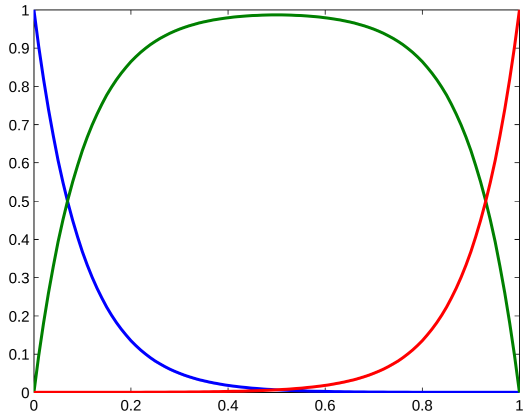
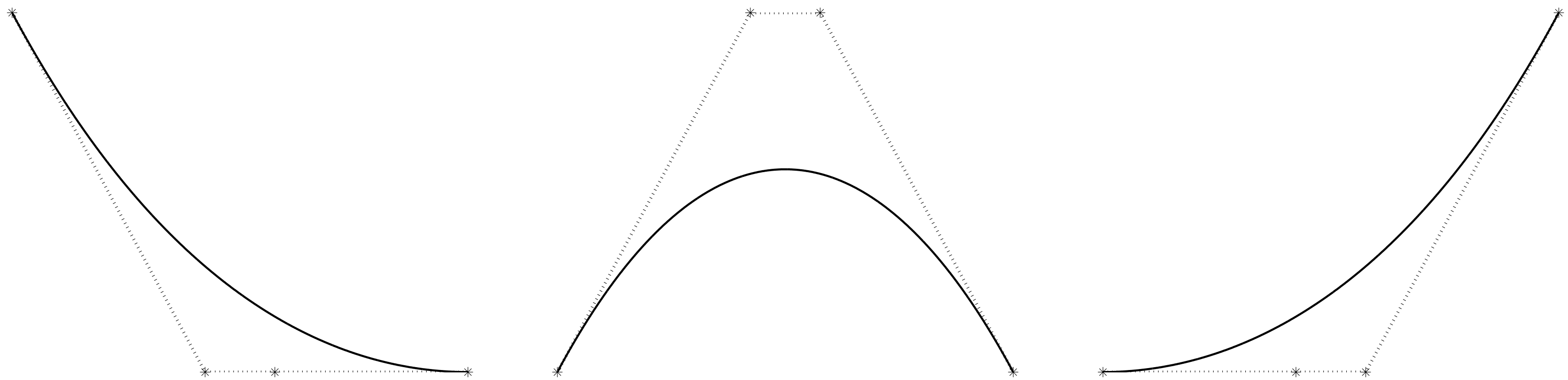






$\omega = 0.1$ $\omega = 1.5$ $\omega = 10$



$\omega = 1.5$

Quadratic Generalized Splines over Triangles

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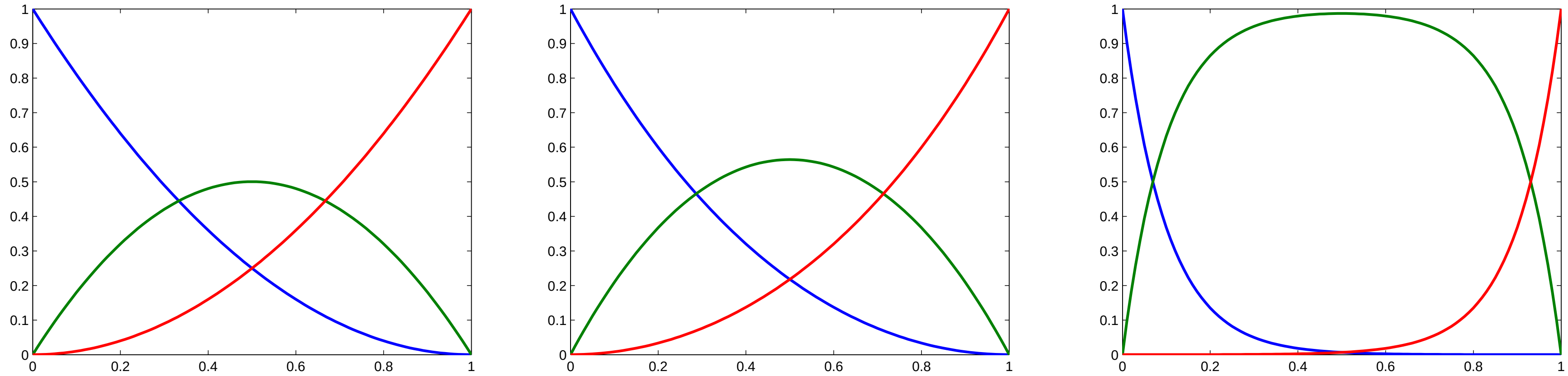
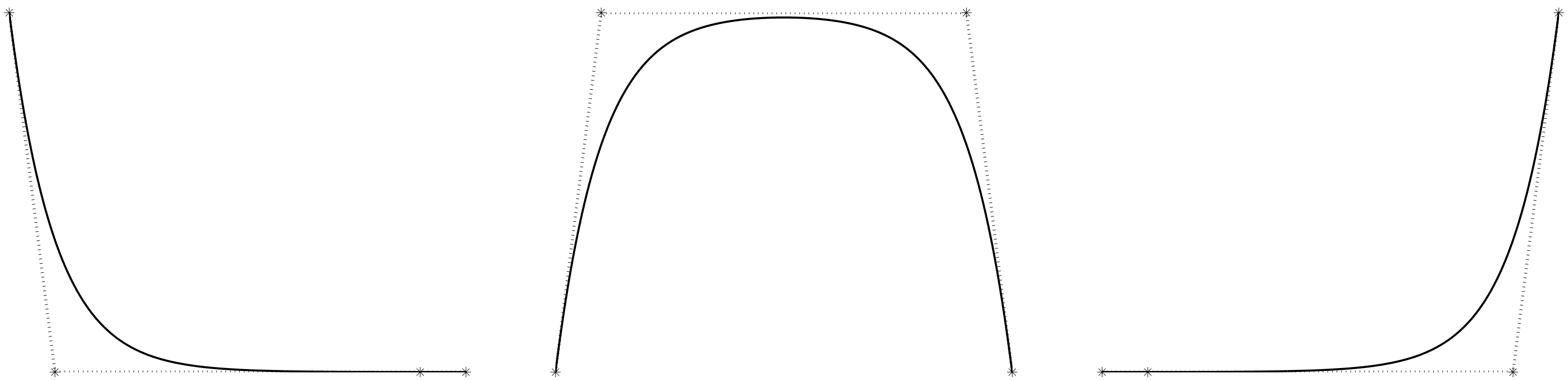


$\omega = 0.1$ $\omega = 1.5$ $\omega = 10$



$\omega = 10$

Quadratic Generalized Splines over Triangles

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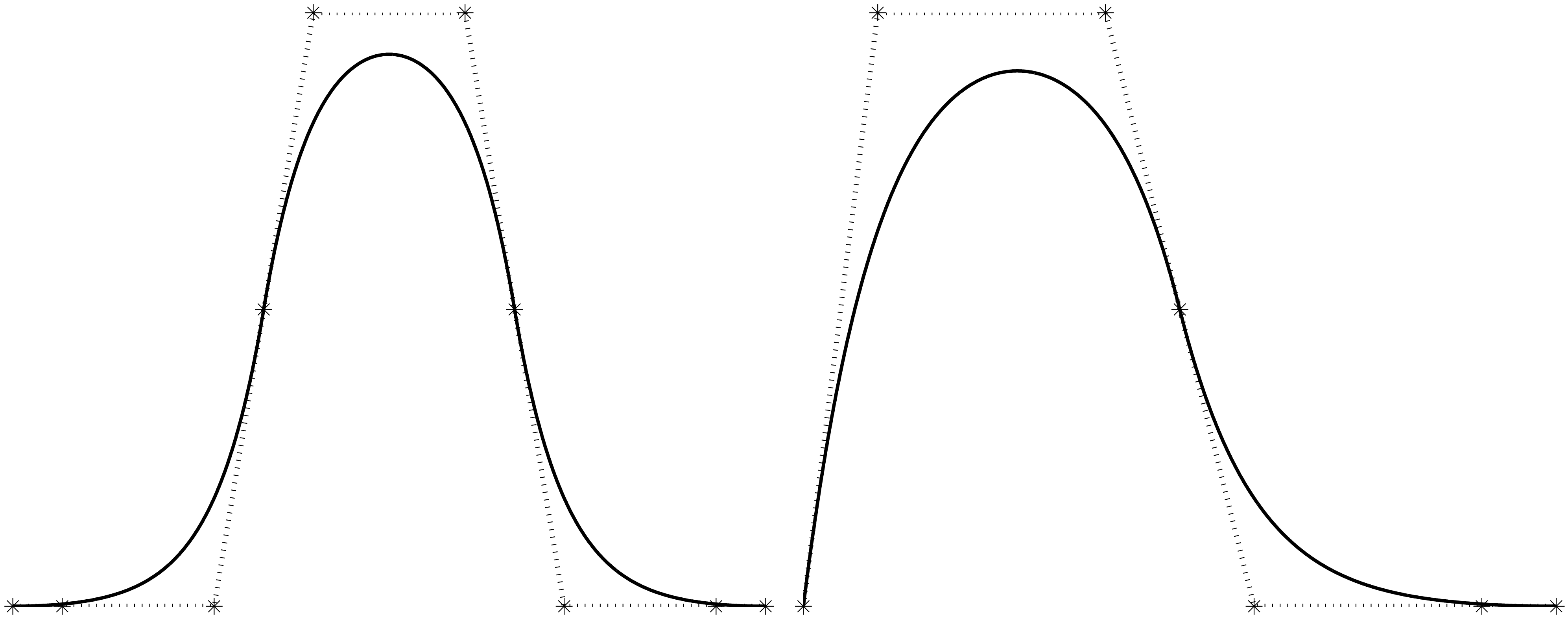
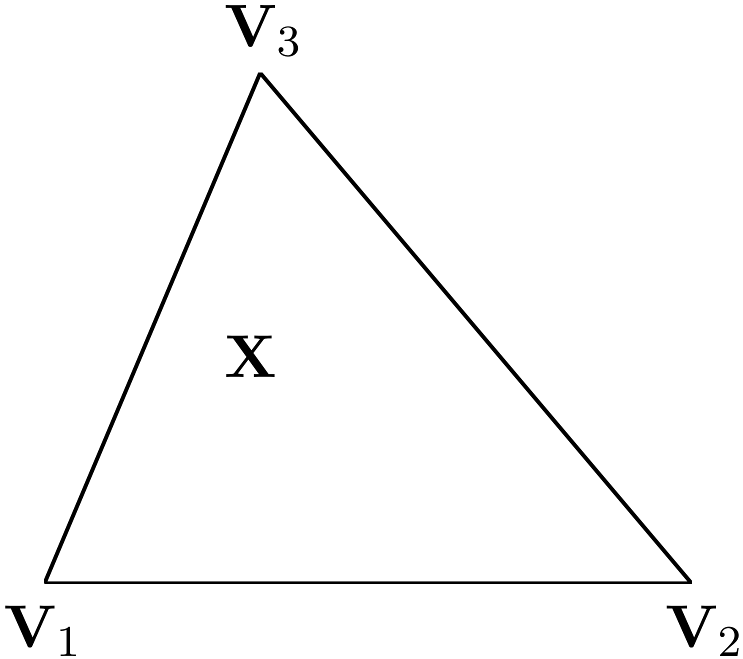
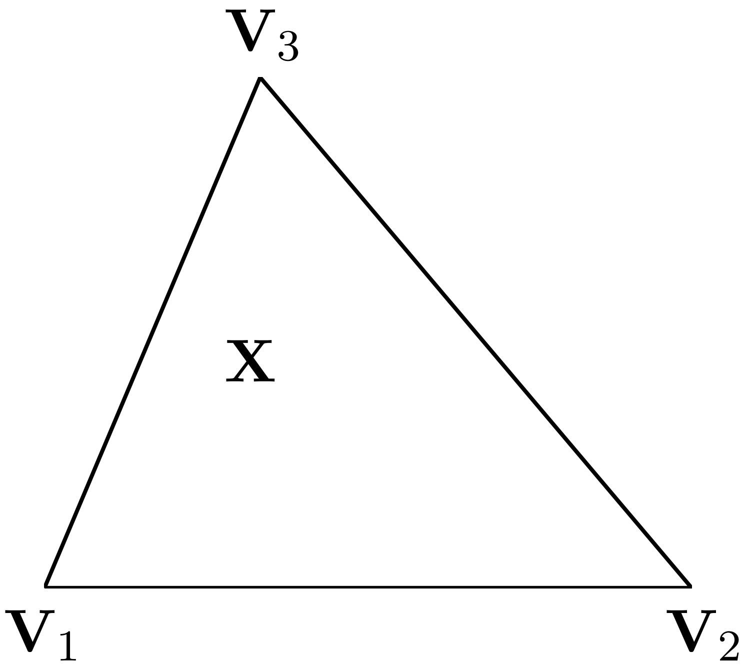


Quadratic Generalized Splines over Triangles



$$\mathbf{X} = \tau_1 \mathbf{V}_1 + \tau_2 \mathbf{V}_2 + \tau_3 \mathbf{V}_3$$

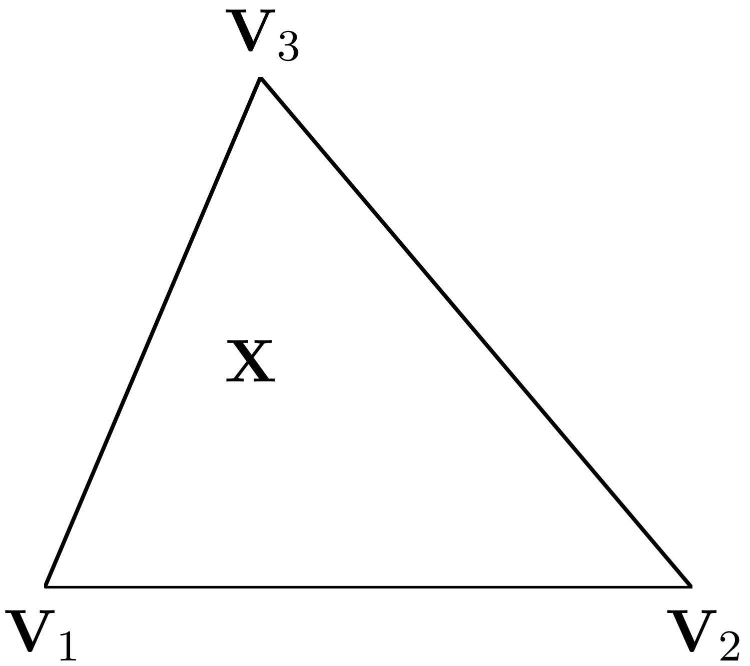
Quadratic Generalized Splines over Triangles



$$\mathbf{X} = \tau_1 \mathbf{V}_1 + \tau_2 \mathbf{V}_2 + \tau_3 \mathbf{V}_3$$

$$< 1, \tau_1, \tau_2, \tau_3 = 1 - \tau_1 - \tau_2, \tau_1^2, \tau_2^2, \tau_3^2 >,$$

Quadratic Generalized Splines over Triangles



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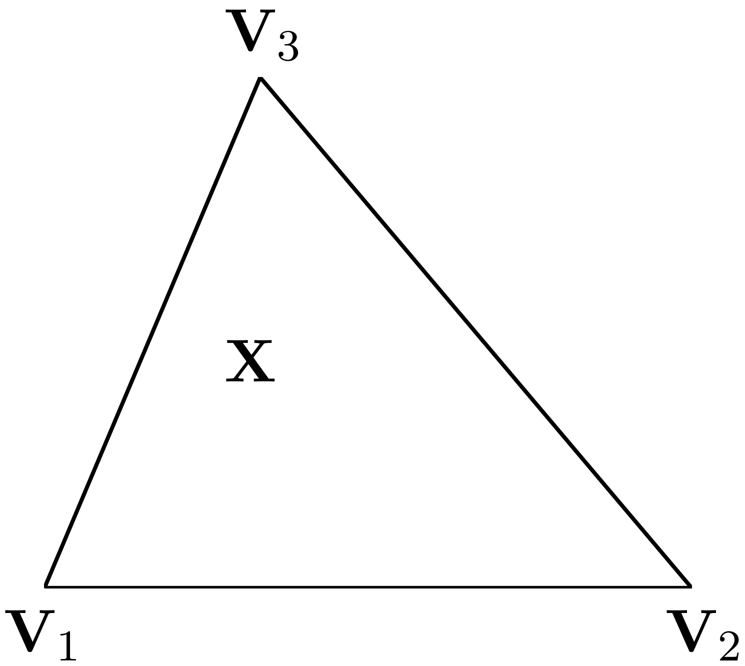
Quadratic Generalized Splines over Triangles



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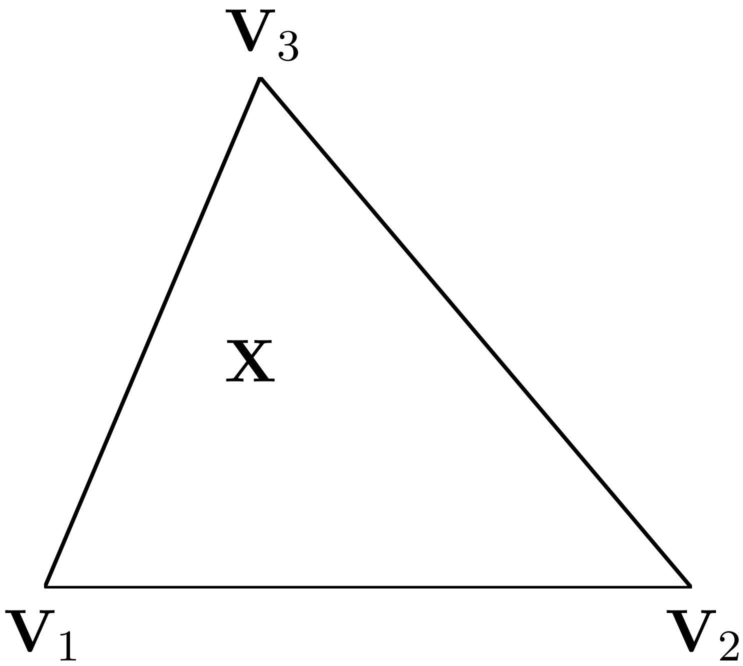
$$\dim(\mathbb{H}_\omega) = 7$$

Quadratic Generalized Splines over Triangles



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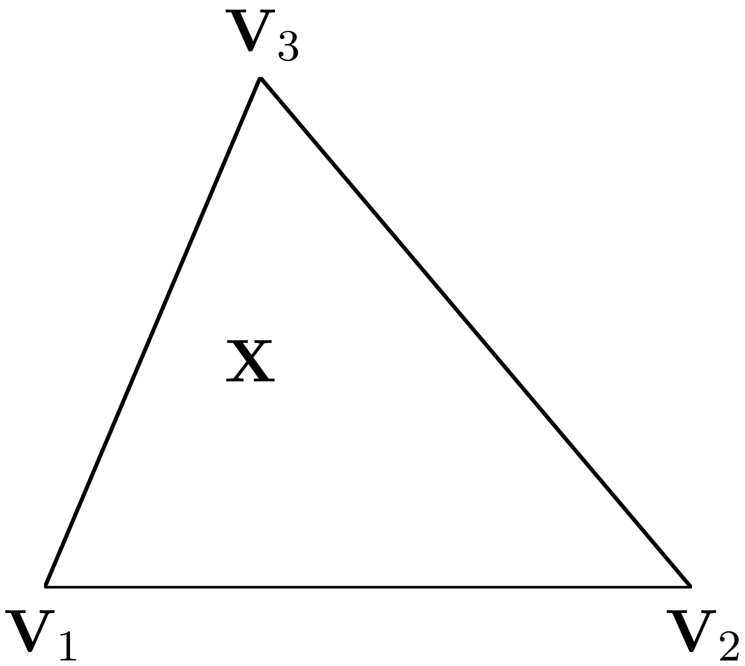
$$\mathbb{H}_{\omega|\tau_3=0} := < 1, \cosh \omega\tau_1, \sinh \omega\tau_1 >,$$

Quadratic Generalized Splines over Triangles



$$\mathbf{X} = \tau_1 \mathbf{V}_1 + \tau_2 \mathbf{V}_2 + \tau_3 \mathbf{V}_3$$

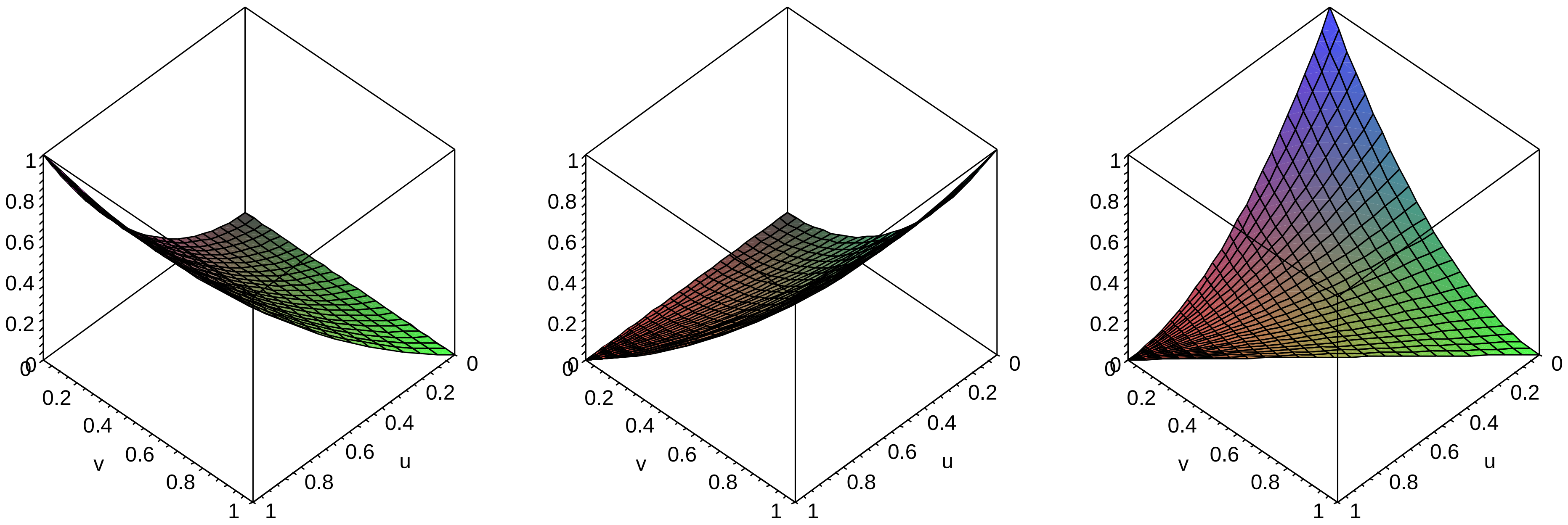
$$B_{200,\omega}(\mathbf{X}) = B_{2,\omega}(\tau_1),$$

$$B_{020,\omega}(\mathbf{X}) = B_{2,\omega}(\tau_2),$$

$$B_{002,\omega}(\mathbf{X}) = B_{2,\omega}(\tau_3)$$

Quadratic Generalized Splines over Triangles

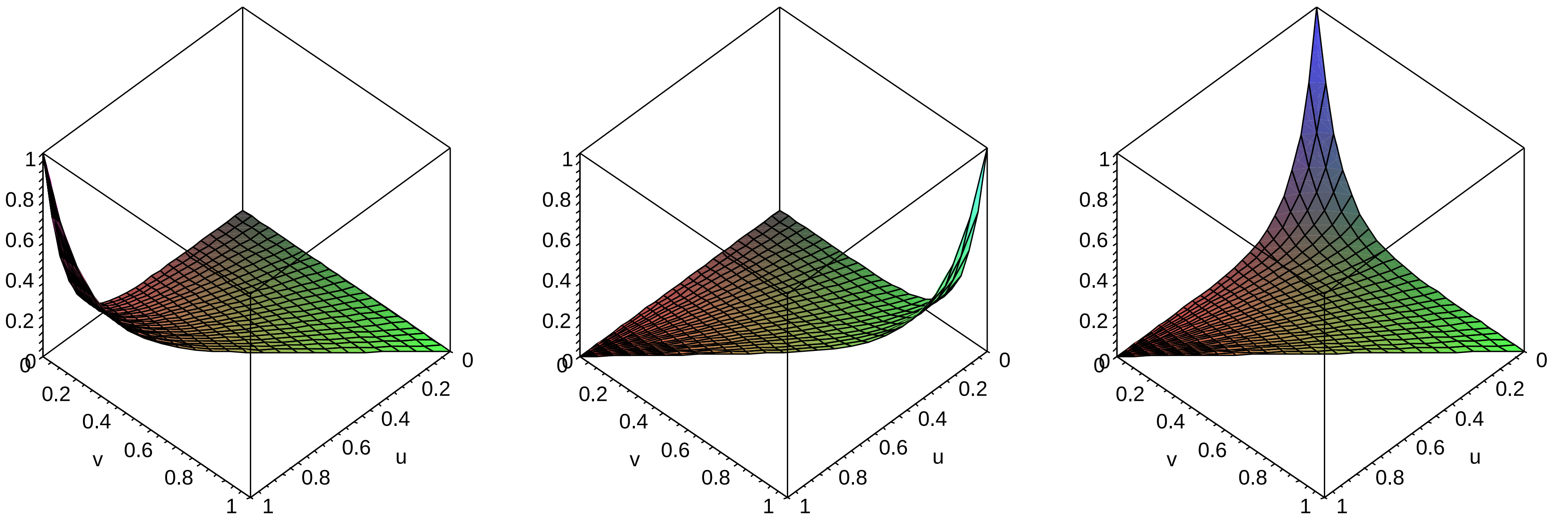
$B_{200,\omega}$ $B_{020,\omega}$ $B_{002,\omega}$



$$\omega = 0.1$$

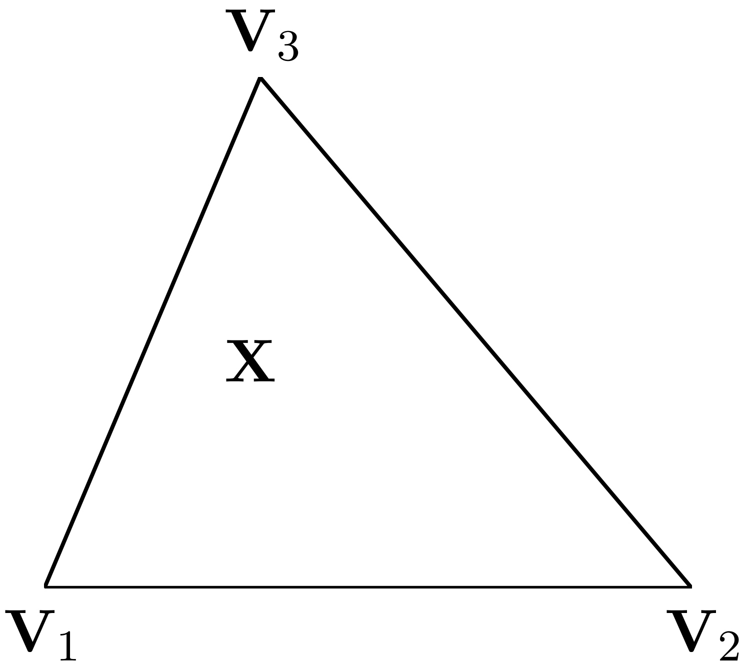
Quadratic Generalized Splines over Triangles

$B_{200,\omega}$ $B_{020,\omega}$ $B_{002,\omega}$



$\omega = 10$

Quadratic Generalized Splines over Triangles



$B_{110,\omega}$???

Quadratic Generalized Splines over Triangles

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Quadratic Generalized Splines over Triangles

$$B_{110,\omega} \quad ???$$

7 suitable interp. conditions to recover edge behavior

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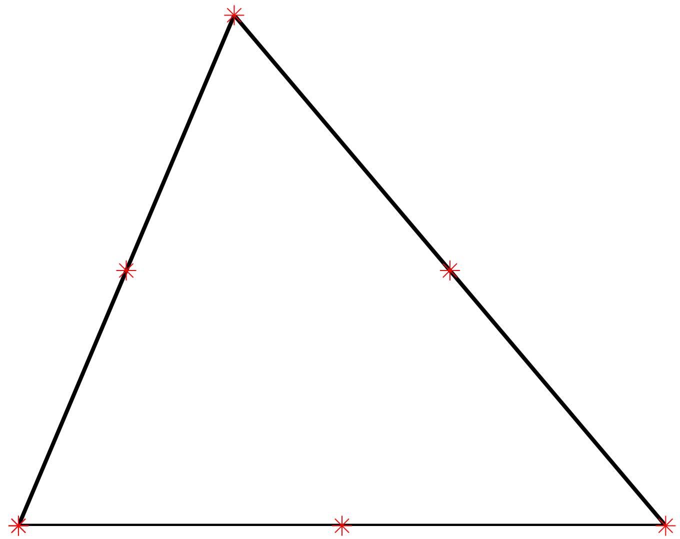
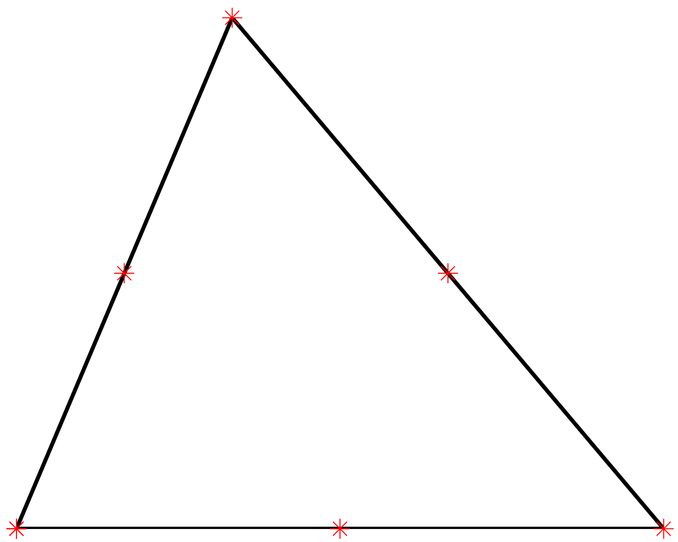


- easy: 6 function values at $*$

Quadratic Generalized Splines over Triangles

$$B_{110,\omega} \quad ???$$

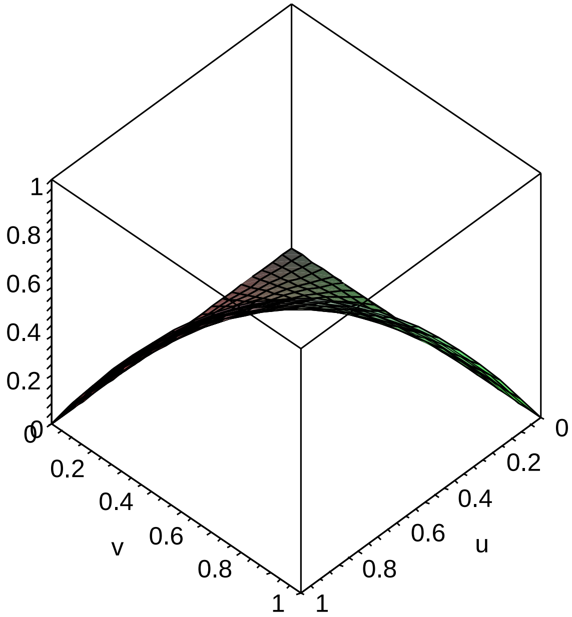
7 suitable interp. conditions to recover edge behavior



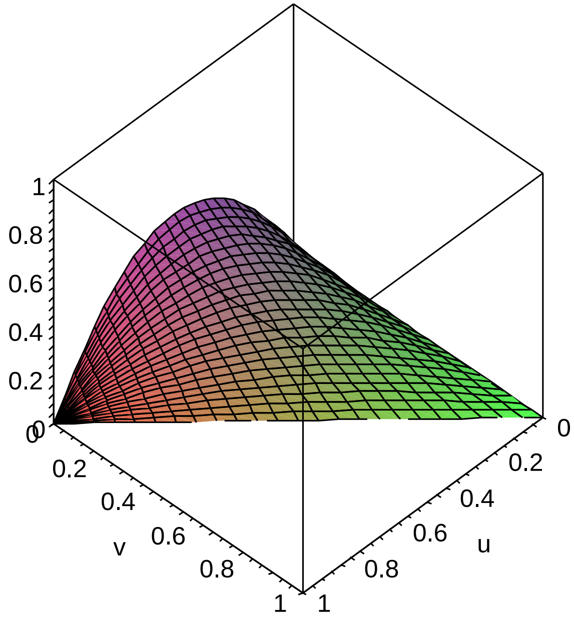
- easy: 6 function values at $*$
- exotic: second derivative at one vertex to mimic the polynomial case

Quadratic Generalized Splines over Triangles

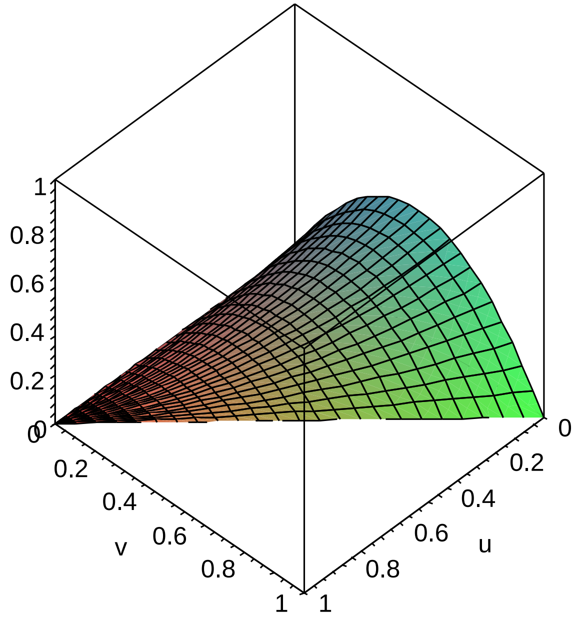
$B_{110,\omega}$



$B_{101,\omega}$



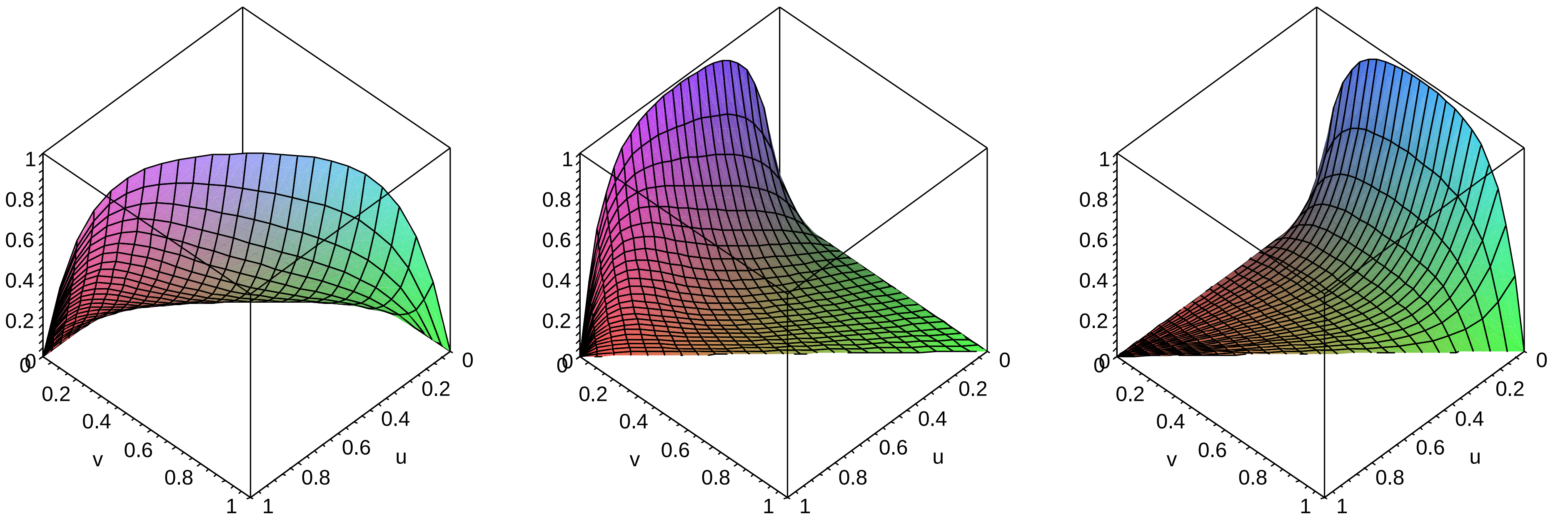
$B_{011,\omega}$



$$\omega = 0.1$$

Quadratic Generalized Splines over Triangles

$B_{110,\omega}$ $B_{101,\omega}$ $B_{011,\omega}$



$$\omega = 10$$

Quadratic Generalized Splines over Triangles

one function still missed

$$B_{111,\omega} \quad ???$$

Quadratic Generalized Splines over Triangles

$$B_{111,\omega} = 1 - \sum_{i+j+k=2} B_{ijk,\omega}$$

Quadratic Generalized Splines over Triangles

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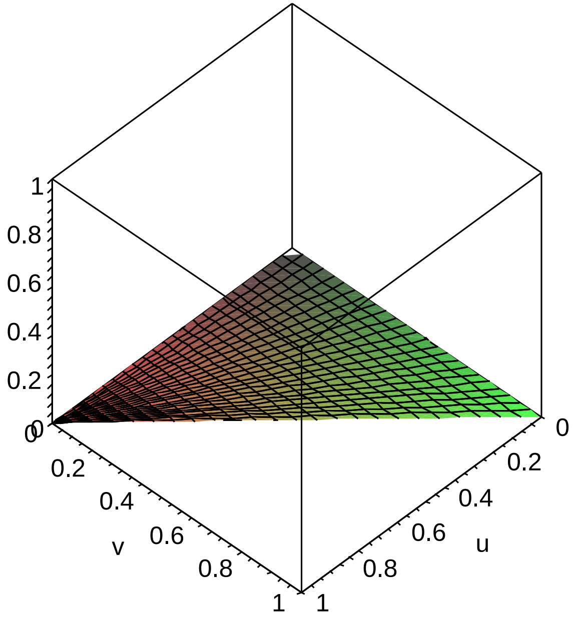
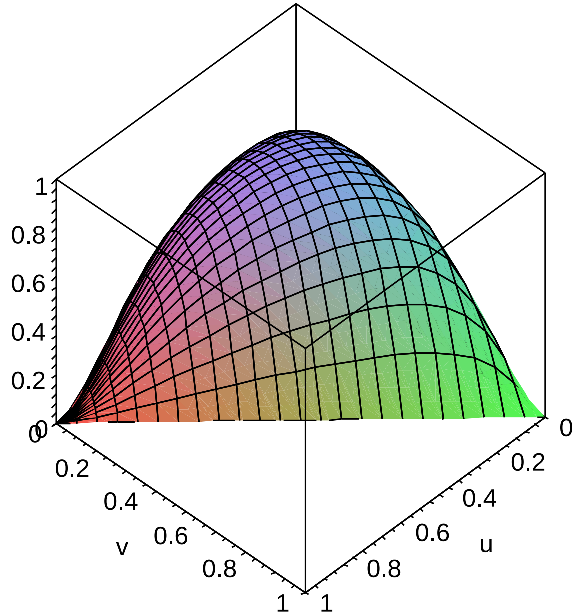


$\omega = .1$

Quadratic Generalized Splines over Triangles

$$B_{111,\omega} = 1 - \sum_{i+j+k=2} B_{ijk,\omega}$$

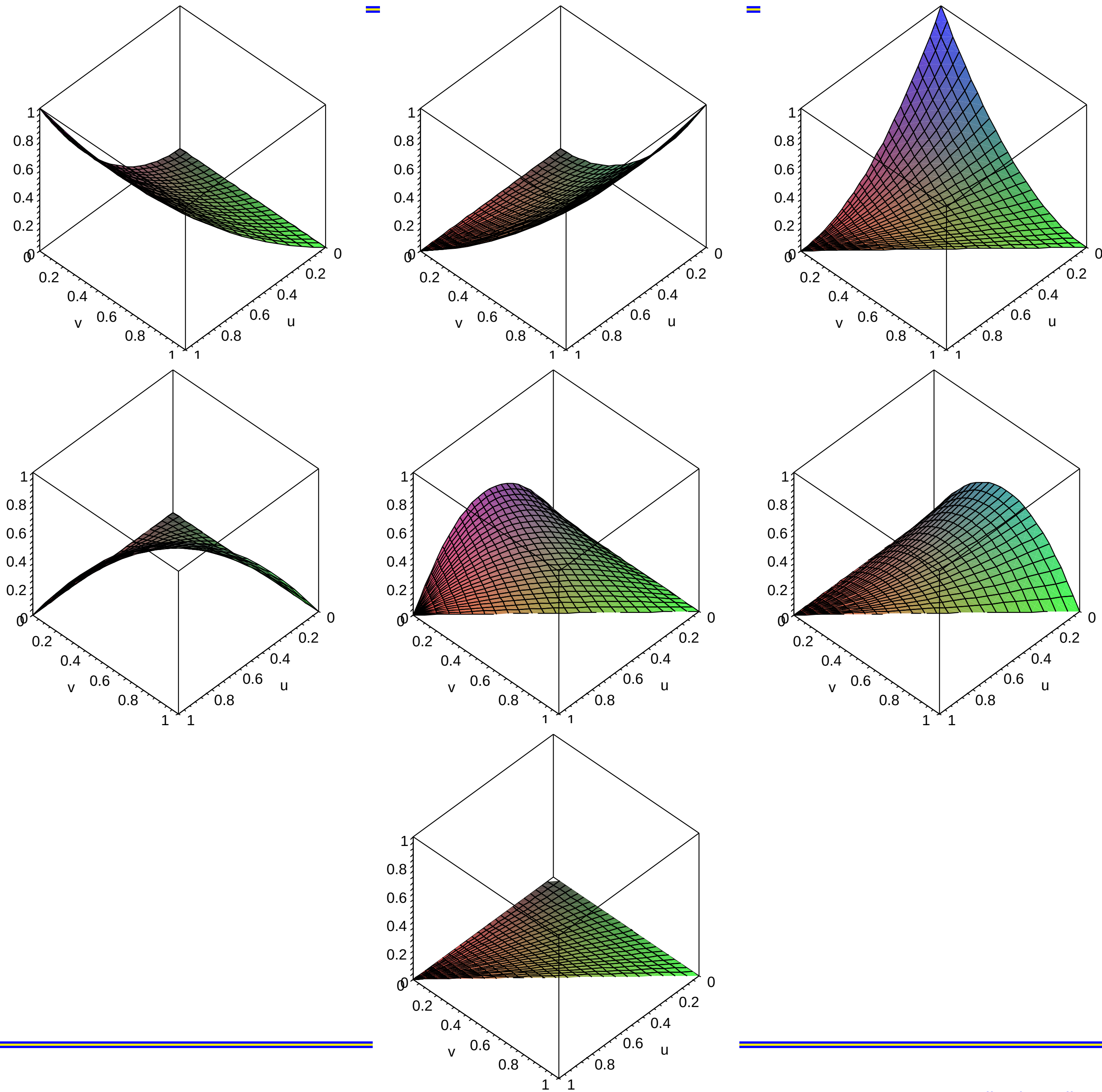


$$\omega = 10$$

Quadratic Generalized Splines over Triangles

- $B_{ijk,\omega} \geq 0$
- partition of unity

Quadratic Generalized Splines over Triangles



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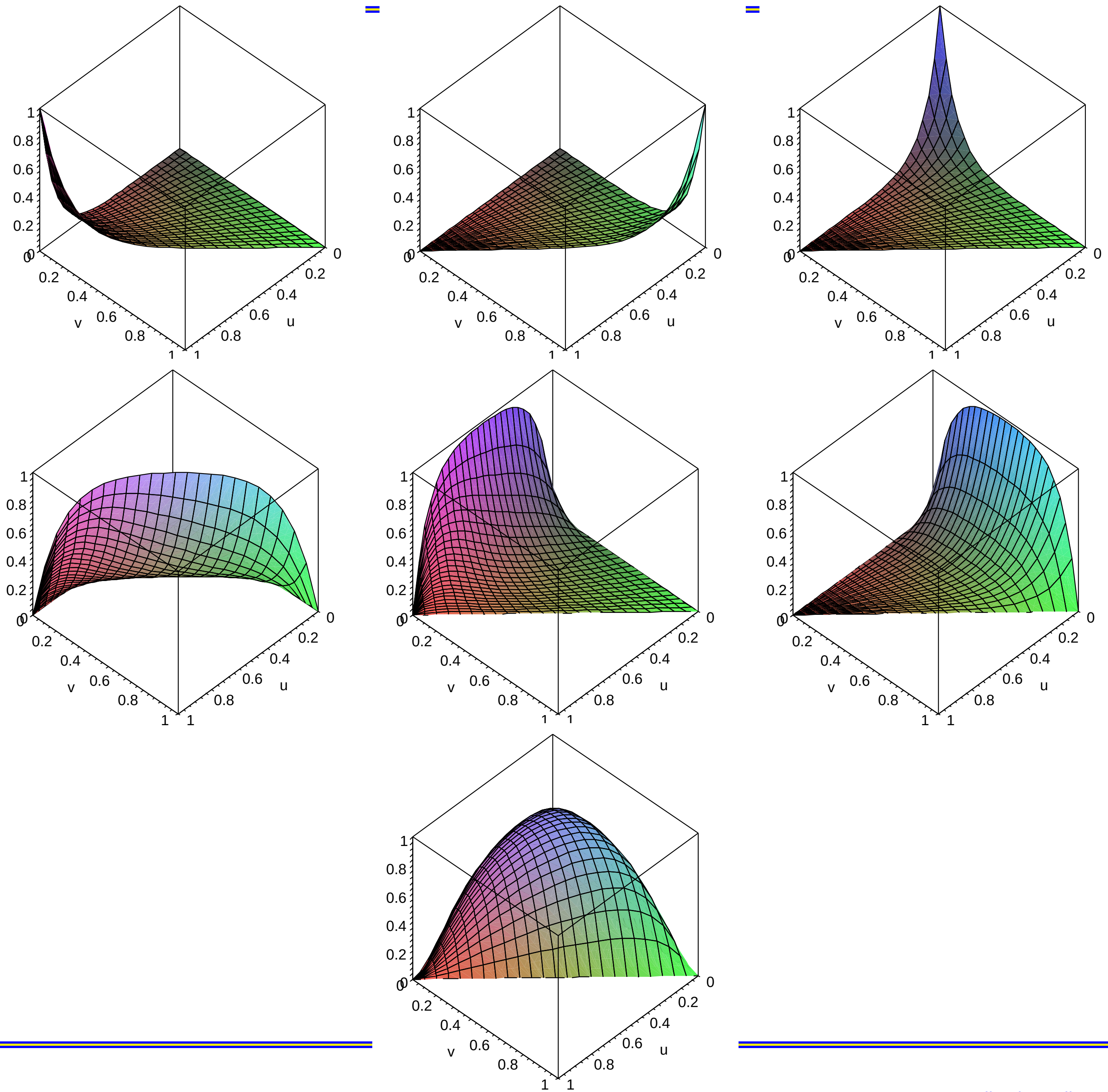
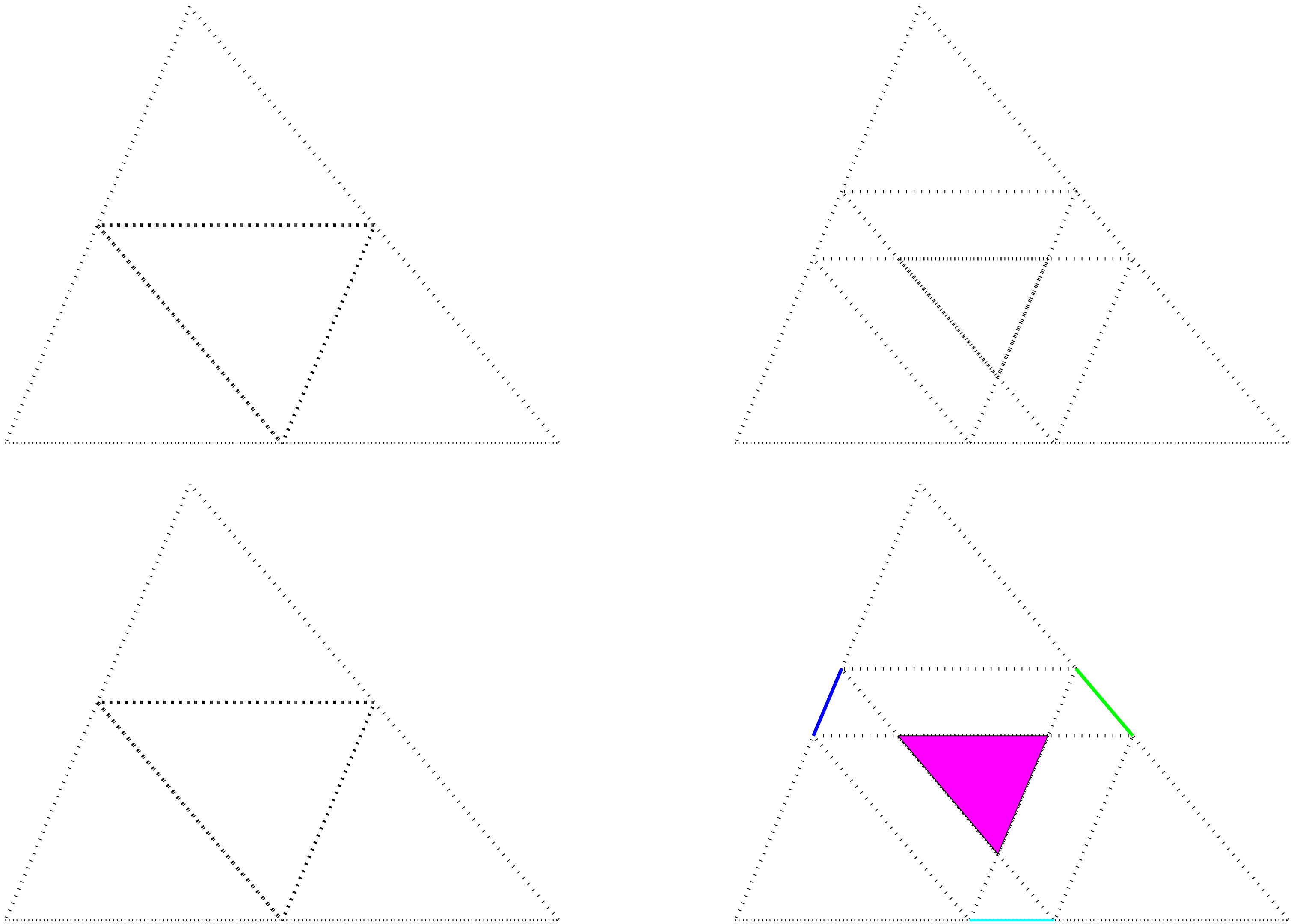


Quadratic G. Splines over Triangles: control net.

NO Greville abscissae

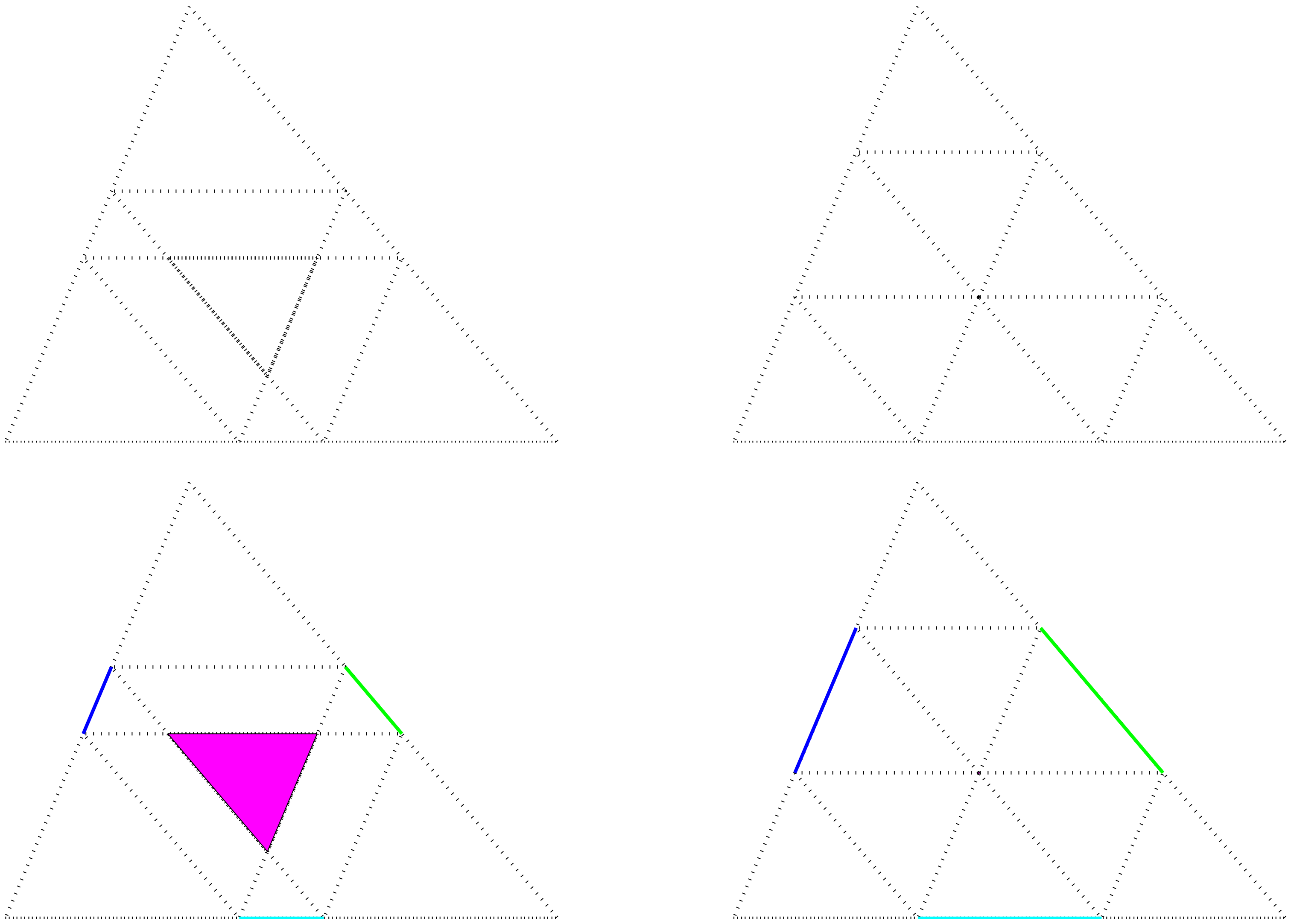
Quadratic G. Splines over Triangles: control net.

$\omega = 0.01$ $\omega = 1.5$



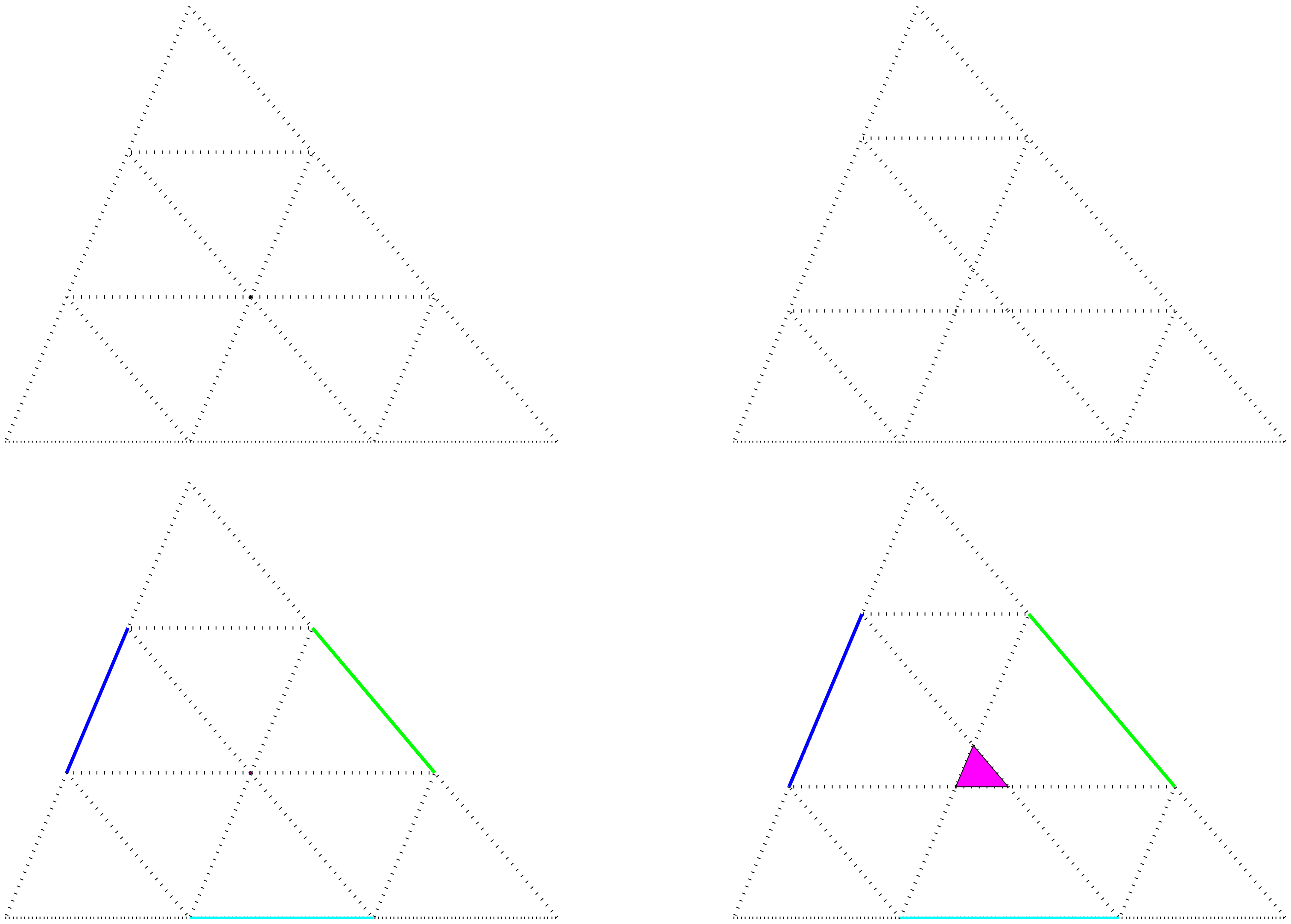
Quadratic G. Splines over Triangles: control net.

$\omega = 1.5$ $\omega = 2.57$



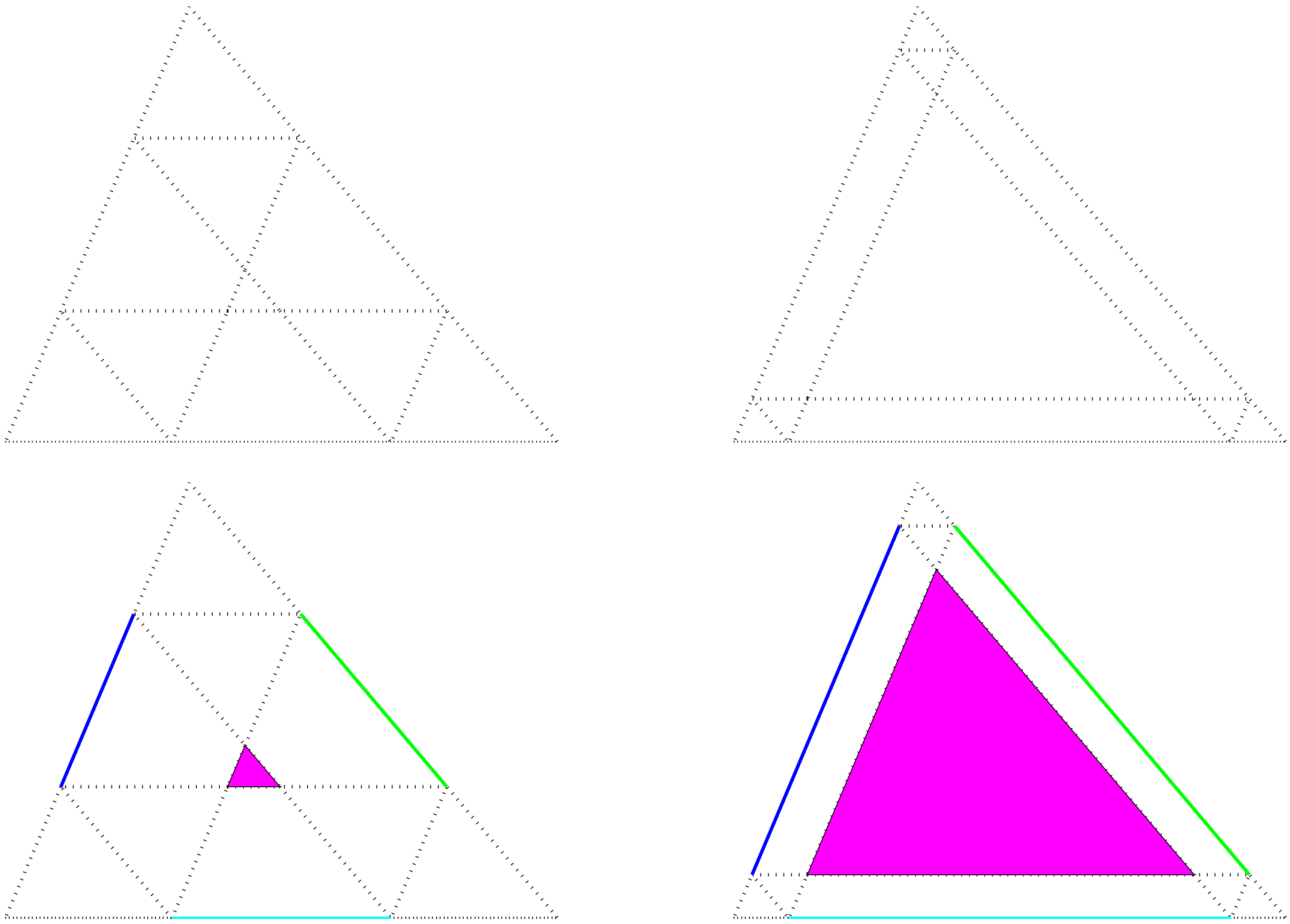
Quadratic G. Splines over Triangles: control net.

$\omega = 2.57$ $\qquad$ $\omega = 3$



Quadratic G. Splines over Triangles: control net.

$\omega = 3$ $\omega = 10$



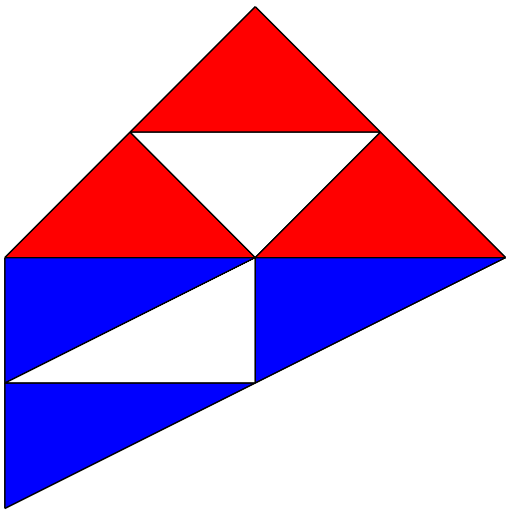
Quadratic G. Splines over Triangles: Smoothness

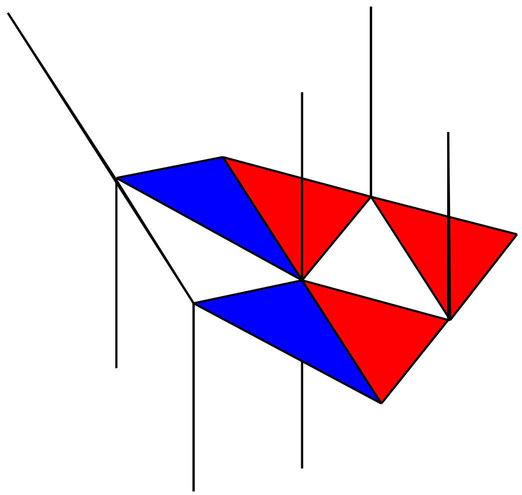
USUAL geometric interpretation

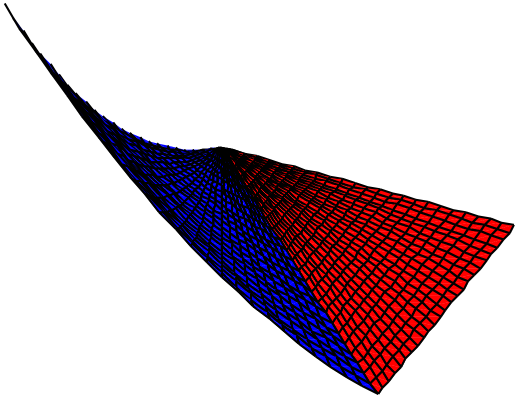
Quadratic G. Splines over Triangles: Smoothness

USUAL geometric interpretation

$$\omega = 0.1$$



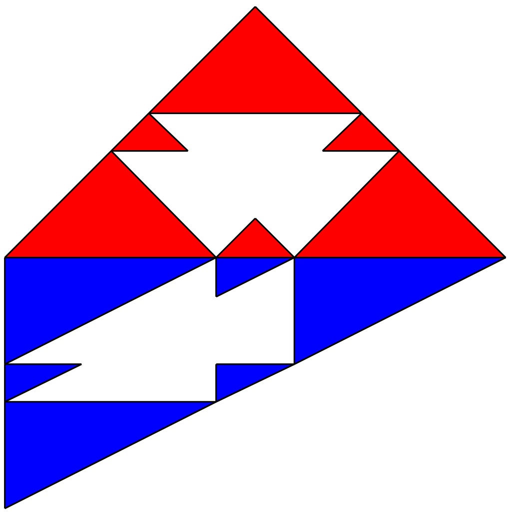


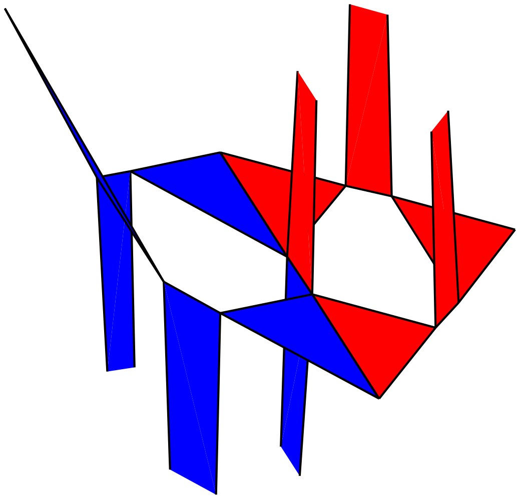


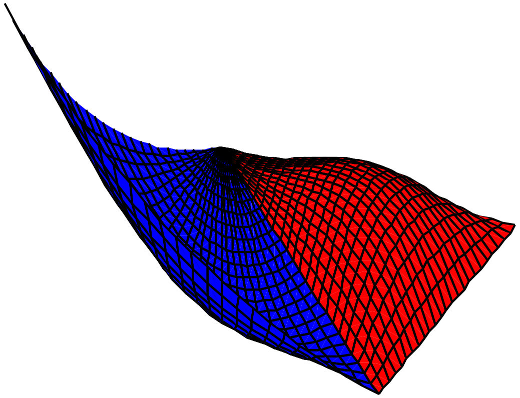
Quadratic G. Splines over Triangles: Smoothness

USUAL geometric interpretation

$$\omega = 1.5$$



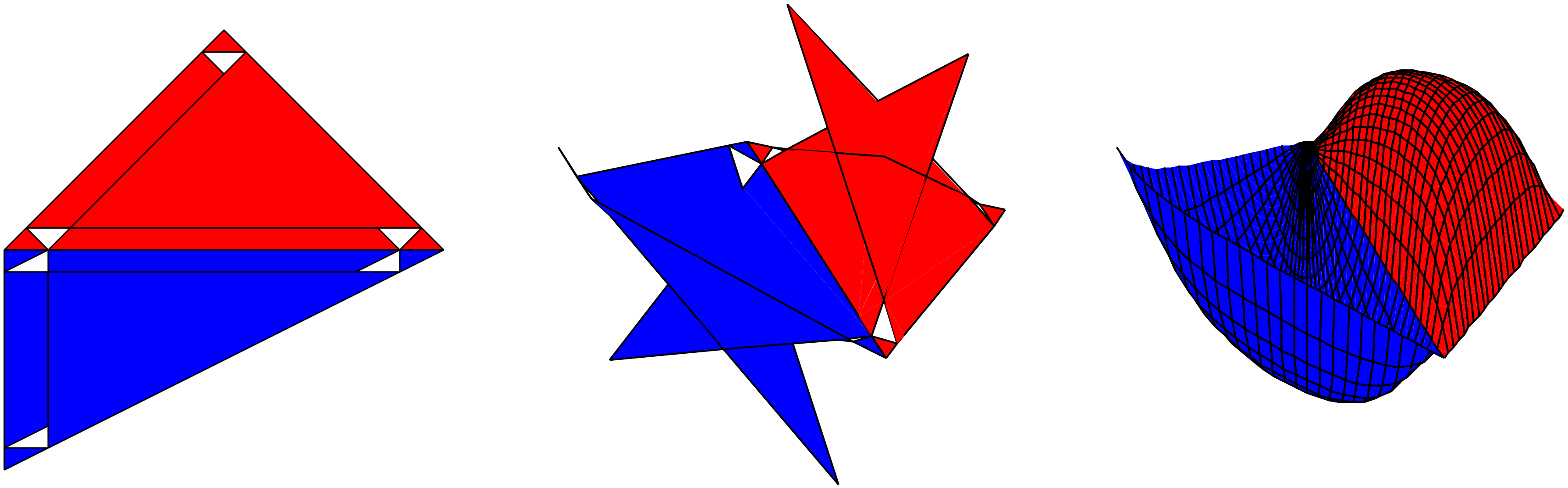




Quadratic G. Splines over Triangles: Smoothness

USUAL geometric interpretation

$$\omega = 10$$



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Conclusions

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  - optimal from geometrical and computational point of view
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  - Bernstein-like representations
  - CAGD applications
  - IgA applications
- Local refinements B-splines/Generalized B-splines
  - Hierarchical bases
  - T-meshes

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Many Thanks!