Extended Hermite Subdivision Schemes

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Overview

1. Vector and Hermite Subdivision schemes,
2. The example of de Rham schemes,
3. Spectral condition, Taylor operators, factorisations and convergence,
4. Extended schemes with B-splines,
5. New extended schemes,
6. What about multivariables?

1. Vector subdivision schemes:

- *Mask:* $\boldsymbol{A} \in \ell^{r\times r}(\mathbb{Z}^s)$ with
  $supp(\boldsymbol{A}) := \{\alpha : \ \boldsymbol{A}(\alpha) \neq 0\} \subset [\sigma, \sigma']^s$,
- *Subdivision Operator:* $S_{\boldsymbol{A}} : \ell^r(\mathbb{Z}^s) \to \ell^r(\mathbb{Z}^s)$
  $$(S_{\boldsymbol{A}}\boldsymbol{c})(\alpha) = \sum_{\beta\in\mathbb{Z}^s} \boldsymbol{A}(\alpha - 2\beta)\boldsymbol{c}(\beta),$$
- *Subdivision Scheme:* $\boldsymbol{f}_0 \in \ell^r(\mathbb{Z}^s)$
  $$\boldsymbol{f}_{n+1}(\alpha) = (S_{\boldsymbol{A}}\boldsymbol{f}_n)(\alpha) = \sum_{\beta\in\mathbb{Z}^s} \boldsymbol{A}(\alpha - 2\beta)\boldsymbol{f}_n(\beta),$$
- *Laurent Polynomial:* $\mathcal{A}^*(\boldsymbol{z}) = \sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{A}(\alpha)\boldsymbol{z}^\alpha$

Convergence:

Let $S_{\boldsymbol{A}}$ be a VSS

- *Convergence:* if for any data $\boldsymbol{f}_0 \in \ell^r(\mathbb{Z}^s)$, there exists a function $\Phi \in \mathcal{C}(\mathbb{R}^s, \mathbb{R}^r)$ such that for any compact $K \subset \mathbb{R}^s$:

$$\lim_{n \to +\infty} \max_{\alpha \in \mathbb{Z}^s, \alpha/2^n \in K} \|\boldsymbol{f}_n(\alpha) - \Phi(\alpha/2^n)\| = 0$$

- *Contractive or Degenerated:* If $\Phi = 0$ for any initial data sequence $\boldsymbol{f}_0$.

The scalar case, $r = s = 1$:

Let S_a be a scalar subd. scheme

- If S_a is convergent then $a^*(z) = (1+z)b^*(z)$ where S_b is a scalar subd. scheme.
- Conversely, let $a^*(z) = (1+z)b^*(z)$. S_a is convergent iff S_b is contractive.
- If $\sum |a(2\alpha)| \leq M$ and $\sum |a(2\alpha+1)| \leq M$ with $M < 1$ then S_a is contractive.
- Let $a^*(z) = \dfrac{(1+z)^{m+1}}{2^m} b^*(z)$ with S_b contractive, then S_a is convergent and the limit function $\varphi \in \mathcal{C}^m(\mathbb{R})$.

The example of B-splines:

Let $\varphi_0(x) = \begin{cases} 1 \text{ if } x \in [0,1[\\ 0 \text{ if } x \notin [0,1[\end{cases}$ and $\varphi_j(x) = \int_{x-1}^{x} \varphi_{j-1}(t)dt$,

then $\varphi_j(x) = \frac{1}{2^j} \sum_{\alpha \in \mathbb{Z}} \binom{j+1}{\alpha} \varphi_j(2x-\alpha)$ for $j \geq 0$. $\varphi_j \in C^{j-1}$.

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$$v(x) := \sum_{\alpha \in \mathbb{Z}} f_0(\alpha)\varphi_j(x-\alpha) \Rightarrow v(x) = \sum_{\alpha \in \mathbb{Z}} f_n(\alpha)\varphi_j(2^n x - \alpha)$$

$$\boxed{f_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}} a_j(\alpha - 2\beta) f_n(\beta)}$$ where $a_j(\alpha) = \frac{1}{2^j}\binom{j+1}{\alpha}$.

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$$a_j^*(z) = \sum_{\alpha \in \mathbb{Z}} a_j(\alpha) z^\alpha = \frac{1}{2^j}(z+1)^{j+1} = \frac{(1+z)^j}{2^{j-1}} c_j^*(z)$$

$c_j^*(z) = \frac{1}{2}(1+z)$, $\sum |c_j(2\alpha)| = 1/2 = \sum |c_j(2\alpha+1)|$.

Convergence and C^{j-1}

Hermite Subdivision Scheme, dim 1

For dimension $s = 1$ with d derivatives, $\boldsymbol{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ the mask, we define the HSS, $H_{\boldsymbol{A}}$ by

$$\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$$

$$\boldsymbol{D}^{n+1}\boldsymbol{f}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta)\,\boldsymbol{D}^n\boldsymbol{f}_n(\beta),$$

where $\boldsymbol{D} = \operatorname{diag}\left[1, \frac{1}{2}, \ldots, \frac{1}{2^d}\right]$.

$\boldsymbol{f}_n(\cdot) = \left[f_n^{(i)}(\cdot)\right]_{i=0,\ldots,d}$ with $f_n^{(i)}(\cdot) \approx \phi_n^{(i)}(\cdot/2^n)$

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- Example of non stationnary VSS, $s=1$, $r=d+1$, $\boldsymbol{f}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}_n(\alpha-2\beta)\,\boldsymbol{f}_n(\beta)$ with $\boldsymbol{A}_n(\cdot) = \boldsymbol{D}^{-(n+1)}\boldsymbol{A}(\cdot)\,\boldsymbol{D}^n$.
- The scheme is interpolant if $\boldsymbol{A}(0) = \boldsymbol{I}$ and $\boldsymbol{A}(2\beta) = \boldsymbol{0}$ for $\beta \neq 0$; in this case $\boldsymbol{f}_{n+1}(2\alpha) = \boldsymbol{f}_n(\alpha)$.

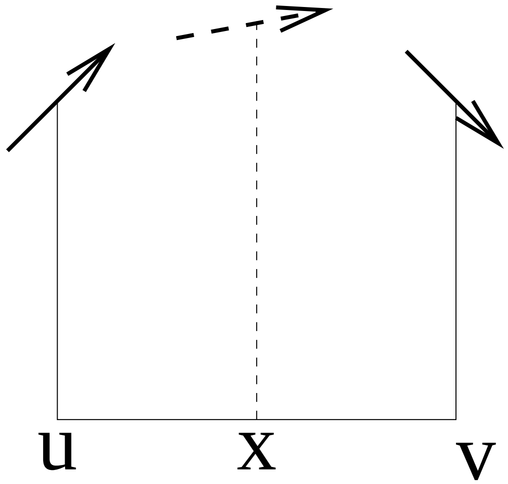
Convergence of HSS

Let $H_{\boldsymbol{A}}$ be Hermite subdivision scheme. The scheme is C^{ℓ}–*convergent* with $\ell \geq d$ if for any initial vector sequence $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ and the corresponding sequence of refinements $\boldsymbol{f}_n$, there exists a vector function $\boldsymbol{\phi} = [\phi^{[i]}]_{i=0,\ldots,d} \in C^{\ell-d}\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ with $\phi^{[0]} \in C^{\ell}(\mathbb{R}, \mathbb{R})$ such that for any compact $K \subset \mathbb{R}$

$$\lim_{n \to \infty} \max_{\alpha \in \mathbb{Z} \cap 2^n K} \|\mathbf{f}_n(\alpha) - \boldsymbol{\phi}\left(2^{-n}\alpha\right)\|_{\infty} = 0$$

and $\phi^{[i]} = \dfrac{d^i \phi^{[0]}}{dx^i}, \; i = 1, \ldots, d\,.$

HC^1, definition:



Cubic Interpolation

$$x = \frac{u+v}{2} \quad , \quad h = v - u,\ p = f'$$

$$\begin{aligned} f(x) &= 1/2\big[f(v) + f(u)\big] \\ &\quad -1/8h\big[p(v) - p(u)\big] \\ p(x) &= 3/2\frac{f(v) - f(u)}{h} \\ &\quad +1/4\big[p(v) + p(u)\big] \end{aligned}$$

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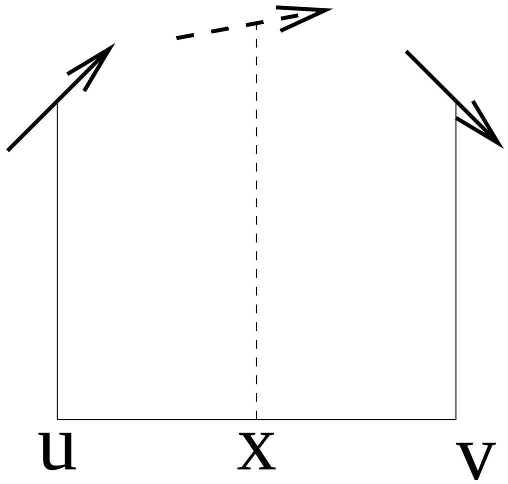

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$$\begin{aligned} f(x) &= 1/2\big[f(v)+f(u)\big] \\ &\quad -1/8h\big[p(v)-p(u)\big] \\ p(x) &= 3/2\frac{f(v)-f(u)}{h} \\ &\quad +1/4\big[p(v)+p(u)\big] \end{aligned}$$

Generalization and iterations:

$h_n = v_n - u_n = h_{n-1}/2$, $x_{n+1} = (u_n + v_n)/2$

$$\begin{aligned} f(x_{n+1}) &= \lambda_1 \frac{f(v_n)+f(u_n)}{2} + \lambda_2 h\big[p(v_n)-p(u_n)\big], \\ p(x_{n+1}) &= \mu_1 \frac{f(v_n)-f(u_n)}{h} + \mu_2\big[p(v_n)+p(u_n)\big]. \end{aligned}$$

Convergence

The scheme is convergent ($\mathcal{C}^1$) if f and p can be extended into continuous functions on $[a, b]$ with $f' = p$. If the scheme is convergent, then $\lambda_1 = 1/2$ and $\mu_1 + 2\mu_2 = 1$.

$$\begin{aligned} f(x_{n+1}) &= \frac{f(v_n) + f(u_n)}{2} + \lambda h_n \left[p(v_n) - p(u_n)\right] \\ p(x_{n+1}) &= (1-\mu)\frac{f(v_n) - f(u_n)}{h_n} + \mu \frac{p(v_n) + p(u_n)}{2} \end{aligned}$$

- Every linear polynomial is reproduced at each step,
- Every quad. pol. is reproduced at each step iff $\lambda = -1/8$,
- Every cub. pol. is reproduced at each step iff $\lambda = -1/8$ and $\mu = -1/2$.

HC^1, a Hermite subdivision scheme:

Given $\lambda, \mu \in \mathbb{R}$, given $\boldsymbol{f}_0 = [f_0^{(0)}, f_0^{(1)}]^T \in \ell^2(\mathbb{Z})$, for $\alpha \in \mathbb{Z}$,

$$
\begin{aligned}
f_{n+1}^{(i)}(2\alpha) &= f_n^{(i)}(\alpha),\ i = 0, 1 \\
f_{n+1}^{(0)}(2\alpha+1) &= 1/2\big[f_n^{(0)}(\alpha+1) + f_n^{(0)}(\alpha)\big] + \lambda 2^{-n}\big[f_n^{(1)}(\alpha+1) - f_n^{(1)}(\alpha)\big] \\
f_{n+1}^{(1)}(2\alpha+1) &= (1-\mu)2^{n}\big[f_n^{(0)}(\alpha+1) - f_n^{(0)}(\alpha)\big] + \mu/2\big[f_n^{(1)}(\alpha+1) + f_n^{(1)}(\alpha)\big].
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\end{aligned}$$

If $f_n^{(0)}(\alpha) = \varphi(2^{-n}\alpha)$ and $f_n^{(1)}(\alpha) = \varphi'(2^{-n}\alpha)$,
cubic interpolation at midpoint for $\lambda = -1/8$, $\mu = -1/2$,
piecewise quadradic interpolation at midpoint for $\lambda = -1/8$, $\mu = -1$.

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$$\boldsymbol{D}^{n+1}\boldsymbol{f}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta)\boldsymbol{D}^n\boldsymbol{f}_n(\beta),$$

$\boldsymbol{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $\boldsymbol{A}(0) = \boldsymbol{D}$, $\boldsymbol{A}(\epsilon)_{\epsilon=\pm 1} = \begin{bmatrix} 1/2 & -\epsilon\lambda \\ -\epsilon(1-\mu)/2 & \mu/4 \end{bmatrix}$, supp$\boldsymbol{A} = \{-1, 0, 1\}$

2. De Rham scheme

From any HSS, $H_{\boldsymbol{A}}$, we define $H_{\overline{\boldsymbol{A}}}$ with the sequence $\overline{f}_n$,

$$\overline{\boldsymbol{f}}_0 = \boldsymbol{f}_0$$

$$\begin{aligned} \boldsymbol{D}^{n+1}\boldsymbol{g}(\beta) &= \sum_{\gamma\in\mathbb{Z}} \boldsymbol{A}(\beta-2\gamma)\boldsymbol{D}^n\overline{\boldsymbol{f}}_n(\gamma),\ \beta\in\mathbb{Z} \\ \boldsymbol{D}^{n+2}\boldsymbol{h}(\alpha) &= \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha-2\beta)\boldsymbol{D}^{n+1}\boldsymbol{g}(\beta),\ \alpha\in\mathbb{Z} \\ \overline{\boldsymbol{f}}_{n+1}(\alpha) &= \boldsymbol{h}(2\alpha+1),\ \alpha\in\mathbb{Z} \end{aligned}$$

then

$$\overline{\boldsymbol{A}}(\alpha) = \boldsymbol{D}^{-1}\sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(2\alpha+1-2\beta)\boldsymbol{A}(\beta), \alpha\in\mathbb{Z}.$$

A Non Interpol. Scheme, $d = 1$

From HC^1, $\overline{\boldsymbol{f}}_0 : \mathbb{Z} \to \mathbb{R}^2$, then, for $n \geq 0$, $\overline{\boldsymbol{f}}_{n+1} : \mathbb{Z} \to \mathbb{R}^2$ is defined by

$$\boldsymbol{D}^{n+1}\overline{\boldsymbol{f}}_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}} \overline{\boldsymbol{A}}(\alpha - 2\beta)\boldsymbol{D}^n\overline{\boldsymbol{f}}_n(\beta), \ \alpha \in \mathbb{Z}$$

where $\boldsymbol{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, supp$\overline{\boldsymbol{A}} = [-2, 1]$. The non zero matrices are

$$\frac{1}{8}\begin{bmatrix} 2 + 4\lambda(1-\mu) & 4\lambda + 2\lambda\mu \\ 4 - 2\mu - 2\mu^2 & \mu^2 + 8\lambda(1-\mu) \end{bmatrix}, \frac{1}{8}\begin{bmatrix} 6 - 4\lambda(1-\mu) & 8\lambda - 2\lambda\mu \\ 4 - 2\mu - 2\mu^2 & 2\mu + \mu^2 - 8\lambda(1-\mu) \end{bmatrix},$$

$$\frac{1}{8}\begin{bmatrix} 6 - 4\lambda(1-\mu) & -8\lambda + 2\lambda\mu \\ -4 + 2\mu + 2\mu^2 & 2\mu + \mu^2 - 8\lambda(1-\mu) \end{bmatrix}, \frac{1}{8}\begin{bmatrix} 2 + 4\lambda(1-\mu) & -4\lambda - 2\lambda\mu \\ -4 + 2\mu + 2\mu^2 & \mu^2 + 8\lambda(1-\mu) \end{bmatrix}.$$

3. Spectral Condition in dimension 1

$$f \in C^d(\mathbb{R}) \longmapsto \boldsymbol{v}_f(\cdot) := \begin{bmatrix} f(\cdot) \\ f'(\cdot) \\ \vdots \\ f^{(d)}(\cdot) \end{bmatrix} \in \ell^{d+1}(\mathbb{Z}).$$

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Spectral condition of order d' if there exist linearly independent polynomials $p_j \in \Pi_j$, $0 \le j \le d'$, such that $S_{\boldsymbol{A}} \boldsymbol{v}_{p_j} = 2^{-j} \boldsymbol{v}_{p_j}$ (eigenvalues and vectors). We can also assume that the polynomials p_j are normalized in such a way that their leading term is $\frac{1}{j!} x^j$.

The spectral condition of order d is equivalent to the sum rule introduced by Bin Han.

Examples:

- For HC^1, the spectral condition of order 1 is satisfied with eigenpolynomials 1 and X. For $\lambda = -1/8$, $\mu = -1$, order 2 and for $\lambda = -1/8$, $\mu = -1/2$ order 3.
- For an interpolating convergent HSS of order d, the spectral condition of order at least d is satisfied with eigenpolynomials $X^j/j!$ for $0 \leq j \leq d$.
- De Rham: If $H_{\boldsymbol{A}}$ of order d satisfies the spectral condition of order ℓ, then the de Rham transform $H_{\overline{\boldsymbol{A}}}$ also satisfies the spectral condition of order ℓ.
- If $H_{\boldsymbol{A}}$ of order d satisfies the spectral condition of order ℓ with corresponding eigenpolynomials $X^j/j!$, $j = 0, \ldots, \ell$, then $H_{\overline{\boldsymbol{A}}}$ also satisfies the spectral condition of order ℓ with eigenpolynomials $(X - 1/2)^j/j!$ for $j = 0, \ldots, \ell$.

Taylor Operators in dim 1:

Partial and Complete Taylor Operators on $\ell^{d+1}(\mathbb{Z})$

$$T_d := \begin{bmatrix} \Delta & -1 & \dots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & 1 \end{bmatrix}, \qquad \widetilde{T}_d := \begin{bmatrix} \Delta & -1 & \dots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & \Delta \end{bmatrix}$$

For $j = 0, \dots, d-1$,

$$\left(T_d \boldsymbol{b}_f\right)_j(\alpha) = \left(\tilde{T}_d \boldsymbol{b}_f\right)_j(\alpha) = f^{(j)}(\alpha+1) - \sum_{k=0}^{d-j} \frac{1}{k!} f^{(j+k)}(\alpha).$$

Taylor Factorization in dim 1

If $\boldsymbol{A} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ satisfies the spectral condition of order d, then there exists two finitely supported mask $\boldsymbol{B}, \tilde{\boldsymbol{B}} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ such that $T_d S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}} T_d$ or $\mathcal{T}_d^*(z)\mathcal{A}^*(z) = 2^{-d}\mathcal{B}^*(z)\mathcal{T}_d^*(z^2)$ and $\tilde{T}_d S_{\boldsymbol{A}} = 2^{-d} S_{\tilde{\boldsymbol{B}}} \tilde{T}_d$ or $\tilde{\mathcal{T}_d}^*(z)\mathcal{A}^*(z) = 2^{-d}\tilde{\mathcal{B}}^*(z)\tilde{\mathcal{T}_d}^*(z^2)$.

$\rightarrow$ Generalization of $a^*(z) = \dfrac{(1+z)^{m+1}}{2^m} b^*(z)$.

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$\rightarrow$ Generalization of $a^*(z) = \dfrac{(1+z)^{m+1}}{2^m} b^*(z)$.

For HC^1,

$$\mathcal{B}^*(z) = \begin{bmatrix} 0 & 0 \\ 2\lambda & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} (1-\mu)/2 & 0 \\ -2\lambda & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 0 \\ -2\lambda & 1 \end{bmatrix} + \begin{bmatrix} (1-\mu)/2 & \mu \\ 2\lambda & 1 \end{bmatrix} z$$

Taylor Fact. of de Rham from HC^1

$H_{\overline{\boldsymbol{A}}}$ satisfies the spectral condition with $p_0(x) = 1$, $p_1(x) = x - 1/2$. A Taylor vector subdivision scheme $S_{\tilde{\overline{\boldsymbol{B}}}}$ is associated with $H_{\overline{\boldsymbol{A}}}$ with supp $\left\{\tilde{\overline{\boldsymbol{B}}}(\alpha)\right\} = [-1, 1]$

$$\mathcal{T}_1^*(z)\overline{\mathcal{A}}^*(z) = \frac{1}{2}\tilde{\overline{\mathcal{B}}}^*(z)\mathcal{T}_1^*(z^2)$$

and

$$\tilde{\overline{\boldsymbol{B}}}(-1) = \frac{1}{4}\begin{bmatrix} 2+4\lambda(1-\mu) & 2\lambda(2+\mu) \\ 0 & \mu^2+8\lambda(1-\mu) \end{bmatrix},$$

$$\tilde{\overline{\boldsymbol{B}}}(0) = \frac{1}{4}\begin{bmatrix} 2\mu+2\mu^2-8\lambda(1-\mu) & -4\lambda(1-\mu)-\mu^2 \\ 0 & 18\mu-16 \end{bmatrix},$$

$$\tilde{\overline{\boldsymbol{B}}}(1) = \frac{1}{4}\begin{bmatrix} -2+4\lambda(1-\mu)+2\mu+2\mu^2 & -20\lambda+14\lambda\mu-3\mu^2-4\mu+4 \\ 4-2\mu-2\mu^2 & 4-2\mu-\mu^2+8\lambda(1-\mu) \end{bmatrix}.$$

Factorization and Convergence in dim 1

Let $H_{\boldsymbol{A}}$ be a Hermite scheme of order d such that the spectral condition of order $\ell \geq d$ is satisfied. Let $S_{\boldsymbol{B}}$ and $S_{\tilde{\boldsymbol{B}}}$ be the associated vector schemes such that $T_d S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}} T_d$ and $\tilde{T}_d S_{\boldsymbol{A}} = 2^{-d} S_{\tilde{\boldsymbol{B}}} \tilde{T}_d$.

- If $S_{\boldsymbol{B}}$ is convergent with limit functions $\Phi \in C^k(\mathbb{R}, \mathbb{R}^d)$, $0 \leq k \leq \ell - d$ of the form $\Phi(\cdot) = [0, \ldots, 0, \varphi^{[d]}(\cdot)]^T$, then the $H_{\boldsymbol{A}}$ is C^{d+k}-convergent.
- If $S_{\tilde{\boldsymbol{B}}}$ is contractive, then the $H_{\boldsymbol{A}}$ is C^d-convergent.

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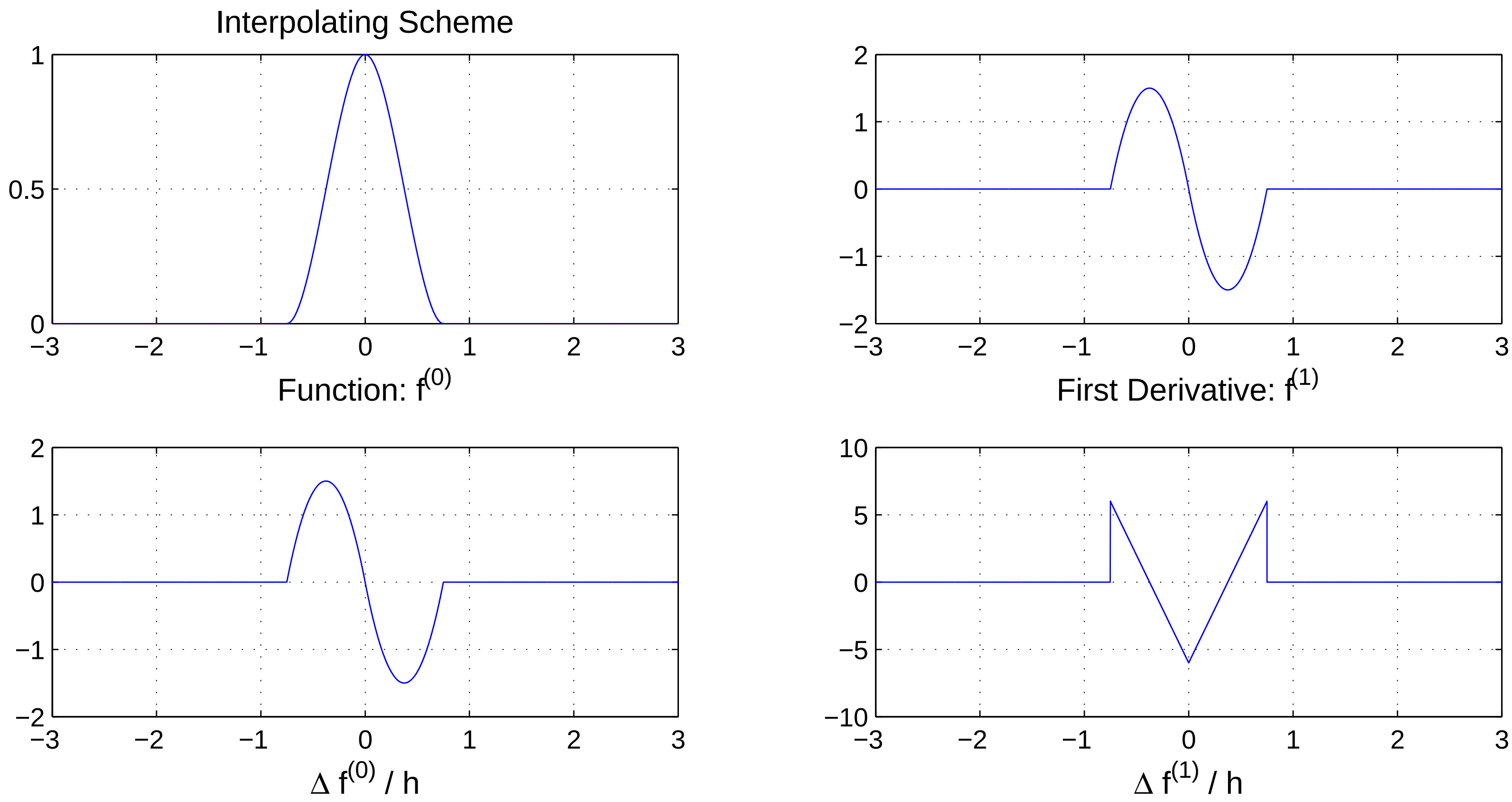
- If $S_{\boldsymbol{B}}$ is convergent with limit functions $\Phi \in C^k(\mathbb{R}, \mathbb{R}^d)$, $0 \leq k \leq \ell - d$ of the form $\Phi(\cdot) = [0, \ldots, 0, \varphi^{[d]}(\cdot)]^T$, then the $H_{\boldsymbol{A}}$ is C^{d+k}-convergent.
- If $S_{\tilde{\boldsymbol{B}}}$ is contractive, then the $H_{\boldsymbol{A}}$ is C^d-convergent.

The question: If a spectral condition of order $\ell > d +$ factorization $+$ contractivity, can we obtain more regularity on the limit function?

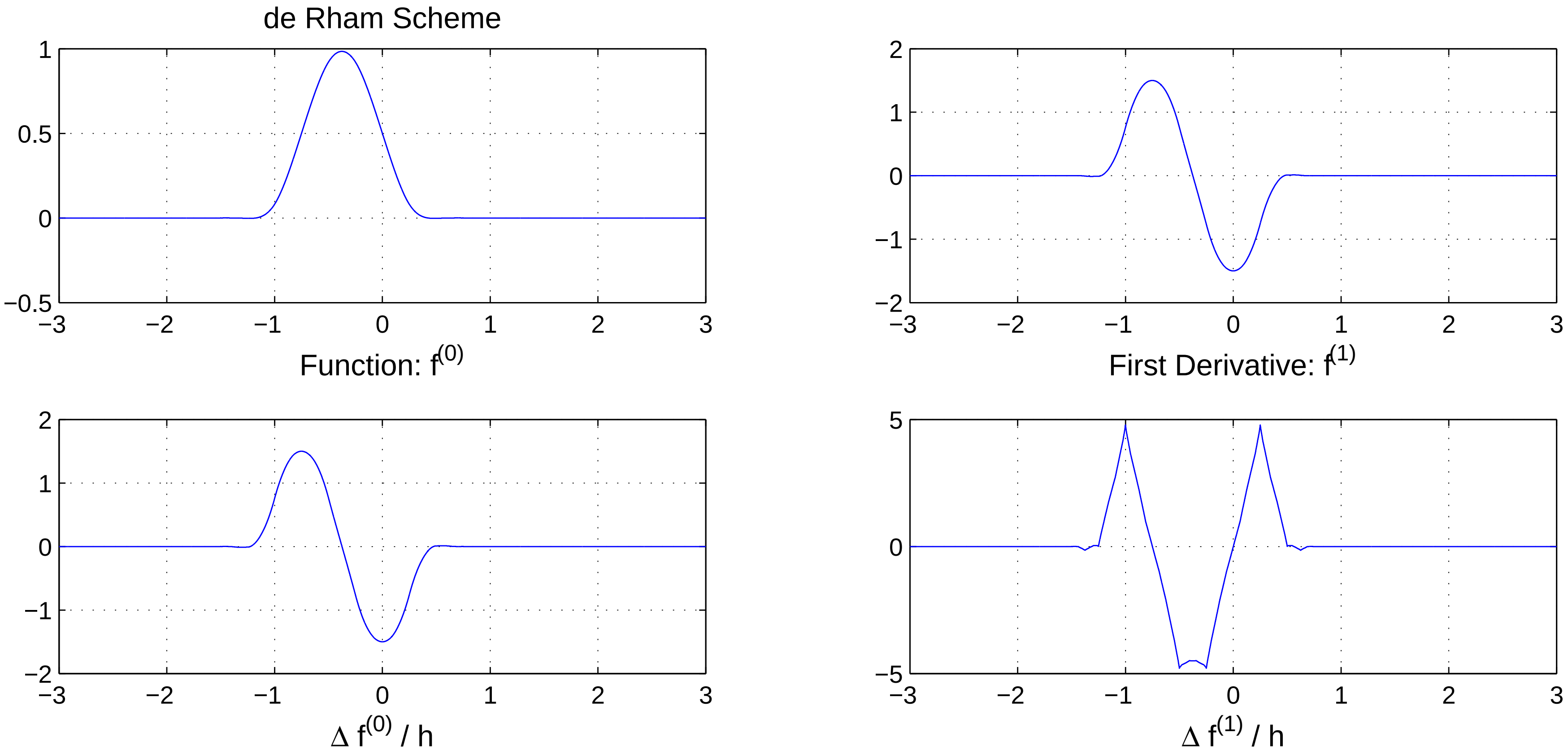
Answer "No" with $H_A = HC^1$

$\lambda = -1/8$ and $\mu = -1/2$ (in HC^1, cubic)



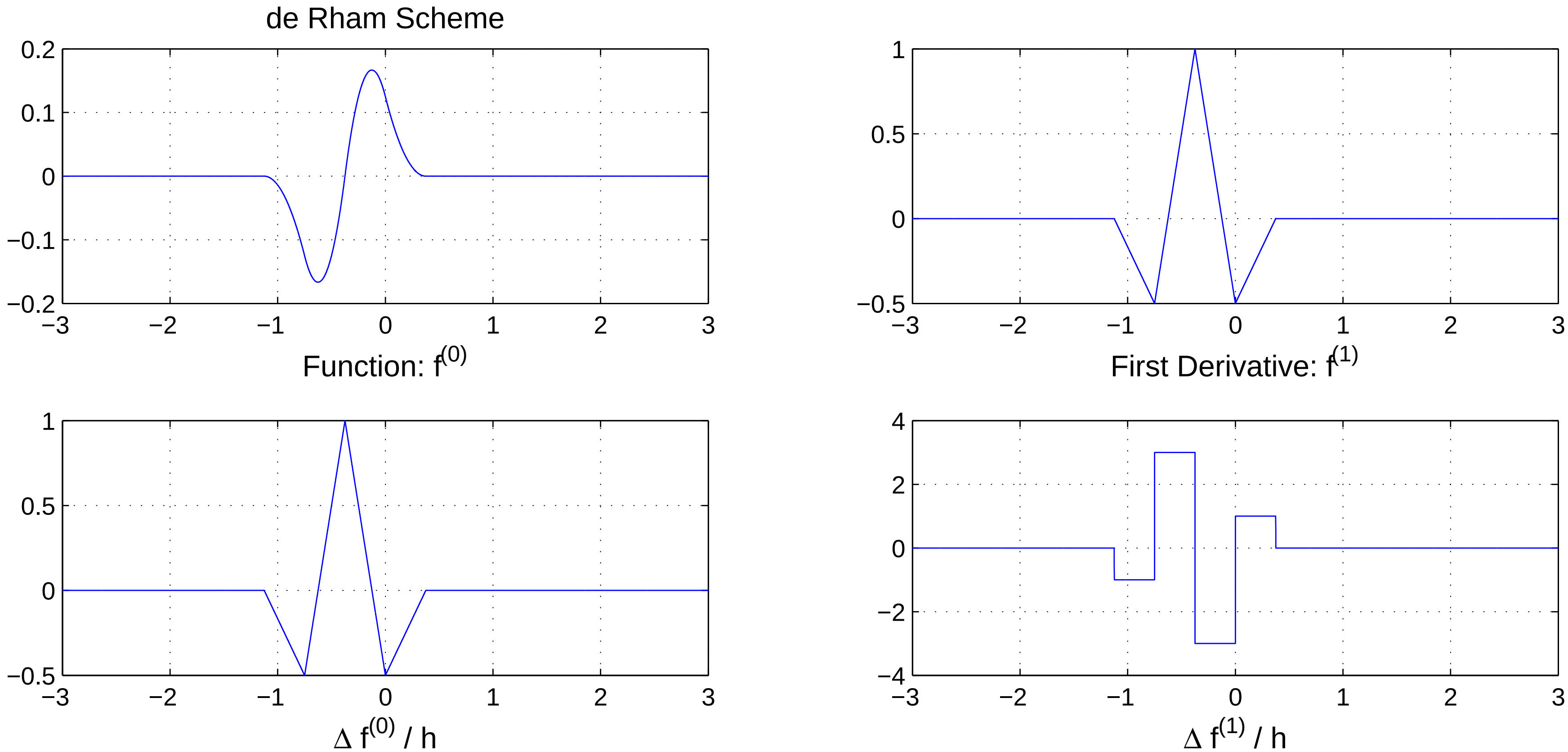
Answer "Yes?" with $H_{\bar{A}}$: de Rham

$\lambda = -1/8$ and $\mu = -1/2$ (from HC^1 cubic)



Answer "No" with $H_{\bar{A}}$: de Rham

$\lambda = -1/8$ and $\mu = -1$ (HC^1 piecewise quadratic)



Factorization and C^2-converg. for $H_{\bar{\boldsymbol{A}}}$

If $\lambda = -1/8$, then HC^1 reproduces polynomials of degree 2 so that the spect. cond. of order 2 is satisfied for $H_{\boldsymbol{A}}$ and $H_{\bar{\boldsymbol{A}}}$.

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$$\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} - 1 \end{bmatrix} \overline{\mathcal{B}}^*(z) = \widetilde{\overline{\mathcal{B}}}^*(z) \begin{bmatrix} 1 & 0 \\ 0 & z^{-2} - 1 \end{bmatrix}.$$

$$\tilde{\bar{\boldsymbol{B}}}(-1) = \frac{1}{16} \begin{bmatrix} 2\mu + 6 & -\mu - 2 \\ -8\mu^2 - 8\mu + 16 & 4\mu^2 + 4\mu - 4 \end{bmatrix}, \quad \tilde{\bar{\boldsymbol{B}}}(0) = \frac{1}{16} \begin{bmatrix} 8\mu^2 + 4\mu + 4 & -4\mu^2 - 2\mu + 2 \\ 0 & 8 \end{bmatrix},$$

$$\tilde{\bar{\boldsymbol{B}}}(1) = \frac{1}{16} \begin{bmatrix} 8\mu^2 + 10\mu - 10 & -4\mu^2 - 5\mu + 8 \\ 8\mu^2 + 8\mu - 16 & -4\mu^2 - 4\mu + 12 \end{bmatrix}.$$

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For $\mu \in [-0.7, 0.424]$, $S_{2\tilde{\bar{\boldsymbol{B}}}}$ is $C^0 \Rightarrow S_{\bar{\boldsymbol{B}}}$ is $C^1 \Rightarrow S_{\bar{\boldsymbol{A}}}$ is C^2

$H_{\boldsymbol{A}} = HC^2$ and $H_{\bar{\boldsymbol{A}}}$: de Rham

$H_{\boldsymbol{A}}$ is interpolating and we choose that it reproduces $\mathbb{P}_3$.
The mask depends on three parameters $\alpha, \beta, \gamma \in \mathbb{R}$.

$$\boldsymbol{A}(-1) = \begin{bmatrix} \frac{1}{2} & \alpha & \frac{-1-8\alpha}{16} \\ \frac{\beta}{2} & \frac{1-\beta}{4} & \frac{2\beta-3}{48} \\ 0 & \frac{\gamma}{4} & \frac{1-\gamma}{8} \end{bmatrix}, \quad \boldsymbol{A}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \quad \boldsymbol{A}(1) = \begin{bmatrix} \frac{1}{2} & -\alpha & \frac{-1-8\alpha}{16} \\ -\frac{\beta}{2} & \frac{1-\beta}{4} & -\frac{2\beta-3}{48} \\ 0 & -\frac{\gamma}{4} & \frac{1-\gamma}{8} \end{bmatrix}.$$

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We apply the de Rham transform: $H_{\bar{\boldsymbol{A}}}$ which satisfies the spectral condition of order 3. Let $S_{\bar{\boldsymbol{B}}}$ and $S_{\tilde{\bar{\boldsymbol{B}}}}$ be the associated vector schemes such that $T_2 S_{\bar{\boldsymbol{A}}} = 2^{-2} S_{\bar{\boldsymbol{B}}} T_2$ and $\widetilde{T}_2 S_{\bar{\boldsymbol{A}}} = 2^{-2} S_{\tilde{\bar{\boldsymbol{B}}}} \widetilde{T}_2$.

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For $\alpha = -\dfrac{5}{32}$, $\gamma = \dfrac{3}{2}$, $\beta \in [1.95, 2.1]$,

$S_{2\tilde{\bar{\boldsymbol{B}}}}$ is $C^0 \Rightarrow S_{\bar{\boldsymbol{B}}}$ is $C^1 \Rightarrow S_{\bar{\boldsymbol{A}}}$ is C^3

4. First extended scheme with B-splines

$$f_{n+1}^0(\alpha) = \sum_{\beta\in\mathbb{Z}} a_j(\alpha - 2\beta) f_n^0(\beta) \text{ where } a_j(\alpha) = \frac{1}{2^j}\binom{j+1}{\alpha}.$$

so that $\Delta f_{n+1}(\alpha) = \sum_\beta \Delta a_j(\alpha - 2\beta) f_n(\beta)$.

If $v(x) = \sum_{\alpha\in\mathbb{Z}} f_n(\alpha)\varphi_j(2^n x - \alpha)$, then

$f_n^0(\cdot) - v(2^{-n}\cdot)$ converges to 0

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Hermite scheme with C^{j-1} convergence:

$$\boldsymbol{D}^{n+1} \boldsymbol{f}_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta) \boldsymbol{D}^n \boldsymbol{f}_n(\beta)$$

with $\boldsymbol{A}(\alpha) = \begin{bmatrix} a_j(\alpha) & 0 \\ \Delta a_j(\alpha - 1) & 0 \end{bmatrix}$ and $\boldsymbol{f}_n(\cdot) = \begin{bmatrix} f^0_n(\cdot) \\ f^1_n(\cdot) \end{bmatrix}$.

Generalization for $j = 4$ and $d = 3$

$$\boldsymbol{A}(\alpha) = \begin{bmatrix} a_4(\alpha) & 0 & 0 & 0 \\ \Delta a_4(\alpha - 1) & 0 & 0 & 0 \\ \Delta^2 a_4(\alpha - 2) & 0 & 0 & 0 \\ \Delta^3 a_4(\alpha - 3) & 0 & 0 & 0 \end{bmatrix}.$$ The scheme is C^3 but the spectral condition of order 2 is not satisfied.

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The scheme is C^3 but the spectral condition of order 2 is not satisfied.

$$\boldsymbol{A}_R(\alpha) = \boldsymbol{R}^{-1}\boldsymbol{A}(\alpha)\boldsymbol{R} \text{ where } \boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which gives

$$\mathcal{A}_R^*(z) = \tfrac{1}{96} \begin{bmatrix} 6(1+z)^5 & 0 & 0 & 0 \\ (1+z)^5(1-z)(11-7z+2z^2) & 0 & 0 & 0 \\ 6(1+z)^5(1-z)^2(2-z) & 0 & 0 & 0 \\ 6(1+z)^5(1-z)^3 & 0 & 0 & 0 \end{bmatrix}$$

The spectral condition of order 3 is satisfied.

Convergence

Let $\widetilde{\boldsymbol{B}}_R$ be defined by $\widetilde{\boldsymbol{T}}_3\boldsymbol{A}_R = 2^{-3}\widetilde{\boldsymbol{B}}_R\widetilde{\boldsymbol{T}}_3$. Then there exists a vector norm such that the corresponding matrix norm satisifies $\sum_{\alpha\in\mathbb{Z}} \|\widetilde{\boldsymbol{B}}_R(2\alpha)\| = \sum_{\alpha\in\mathbb{Z}} \|\widetilde{\boldsymbol{B}}_R(2\alpha+1)\| = 5/12 < 1$.

Therefore

- The operator $S_{\widetilde{\boldsymbol{B}}_R}$ is contractive,
- The Hermite scheme $H_{\boldsymbol{A}_R}$ is C^3,
- Reprove the known result that $H_{\boldsymbol{A}}$ is also C^3.

5. New extended scheme

Idea: Let $\varphi \in C^{d+1}(\mathbb{R})$, then

$$\begin{aligned} \varphi^{(d+1)}(2^{-(n+1)}2\alpha) &\simeq \frac{\varphi^{(d)}(2^{-n}(\alpha+1)) - \varphi^{(d)}(2^{-n}(\alpha-1))}{2^{-n+1}} \\ \varphi^{(d+1)}(2^{-(n+1)}(2\alpha+1)) &\simeq \frac{\varphi^{(d)}(2^{-n}(\alpha+1)) - \varphi^{(d)}(2^{-n}(\alpha))}{2^{-n}} \end{aligned}$$

Extension of the Hermite subdivision scheme:

$$\begin{aligned} f_{n+1}^{(d+1)}(2\alpha) &= \frac{f_n^{(d)}(\alpha+1) - f_n^{(d)}(\alpha-1)}{2^{-n+1}} \\ f_{n+1}^{(d+1)}(2\alpha+1) &= \frac{f_n^{(d)}(\alpha+1) - f_n^{(d)}(\alpha)}{2^{-n}} \end{aligned}$$

Extended mask

Let $H_{\boldsymbol{A}}$ be a Hermite scheme of order d i.e. $\boldsymbol{A}(\alpha) \in \mathbb{R}^{(d+1)\times(d+1)}$.
The extended mask $\boldsymbol{A}_+(\alpha) \in \mathbb{R}^{(d+2)\times(d+2)}$:

$$\boldsymbol{A}_+(-2) = \begin{bmatrix} & \boldsymbol{A}(-2) & & \mathbf{0} \\ 0 & \mathbf{0} & 1/2^{d+2} & 0 \end{bmatrix}, \quad \boldsymbol{A}_+(-1) = \begin{bmatrix} & \boldsymbol{A}(-1) & & \mathbf{0} \\ 0 & \mathbf{0} & 1/2^{d+1} & 0 \end{bmatrix},$$

$$\boldsymbol{A}_+(1) = \begin{bmatrix} & \boldsymbol{A}(1) & & \mathbf{0} \\ 0 & \mathbf{0} & -1/2^{d+1} & 0 \end{bmatrix}, \quad \boldsymbol{A}_+(2) = \begin{bmatrix} & \boldsymbol{A}(2) & & \mathbf{0} \\ 0 & \mathbf{0} & -1/2^{d+2} & 0 \end{bmatrix},$$

$$\boldsymbol{A}_+(\alpha) = \begin{bmatrix} & \boldsymbol{A}(\alpha) & & \mathbf{0} \\ 0 & \mathbf{0} & 0 & 0 \end{bmatrix} \text{ for } \alpha \notin \{-2,-1,1,2\}.$$

If $H_{\boldsymbol{A}}$ satisfies the spectral condition of order $d+1$, then $H_{\boldsymbol{A}_+}$ satisfies the spectral condition of order $d+1$.

New extension for B-Splines, $j = 5$

$$\mathcal{A}_+^*(z) = \frac{1}{16}\begin{bmatrix} (z+1)^5 & 0 \\ 4(z^{-2}-1)*(z+1)^2 & 0 \end{bmatrix}$$

$$\mathcal{B}_+{}^*(z) =$$

$$\begin{bmatrix} \frac{(z-1)*(z+1)^2*(z^2+3*z+4)}{8} & \frac{(z-1)*(z+1)^2*(z^2+3*z+4)}{8} \\ \frac{(z+1)^2}{2} & \frac{(z+1)^2}{2} \end{bmatrix}$$

$$\widetilde{\mathcal{B}_+}^*(z) =$$

$$\begin{bmatrix} \frac{(z-1)*(z+1)^2*(z^2+3*z+4)}{8} & -\frac{(z^2*(z+1)*(z^2+3*z+4)}{8} \\ -\frac{(z-1)*(z+1)^2}{2*z} & \frac{z*(z+1)}{2} \end{bmatrix}$$

Double extension

$$\mathcal{A}^{*}_{++}(z) = \frac{1}{16}\begin{bmatrix} (z+1)^5 & 0 & 0 \\ 4(z^{-2}-1)*(z+1)^2 & 0 & -10(z+1) \\ 0 & 4(z^{-2}-1)*(z+1)^2/4 & -4(z+1) \end{bmatrix};$$

$$\widetilde{\mathcal{B}_{++}}^{*}(z) =$$

$$\begin{bmatrix} \frac{(z-1)*(z+1)^2*(z^2+3*z+4)}{4} & -\frac{(z+1)*(z^4+3*z^3+4*z^2+2*z+2)}{4} & \dots \\ -\frac{(z-1)*(z+1)^2}{z} & -z-1 & \dots \\ 0 & -\frac{(z-1)*(z+1)^2}{z} & \dots \end{bmatrix}$$

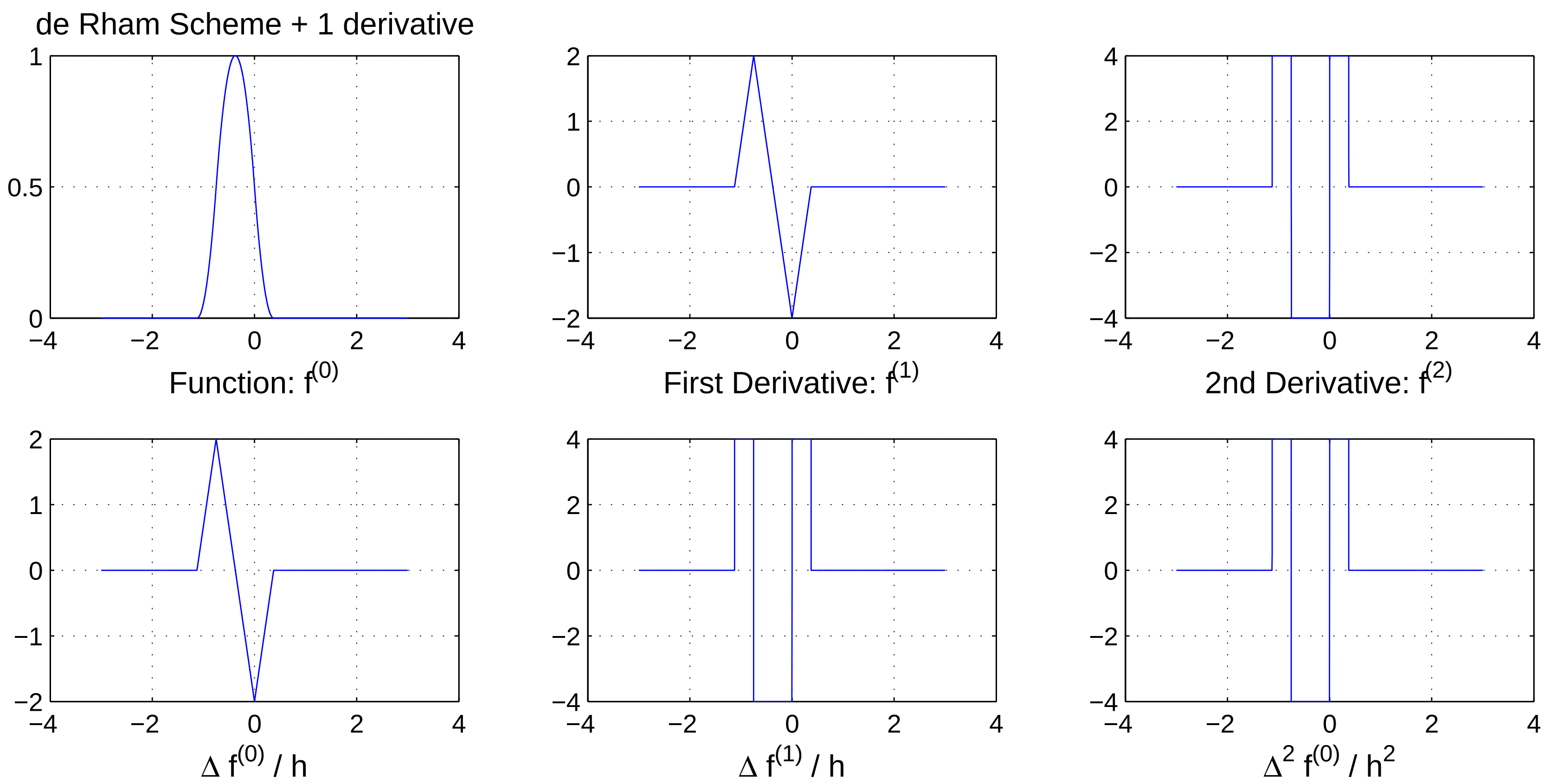
$$\begin{matrix} \dots & \frac{z^2*(z^3+4*z^2+9*z+16)}{8} \\ \dots & \frac{(z-4)*z}{2} \\ \dots & z^2 \end{matrix}\Bigg]$$

Extension of De Rham (from HC^1)

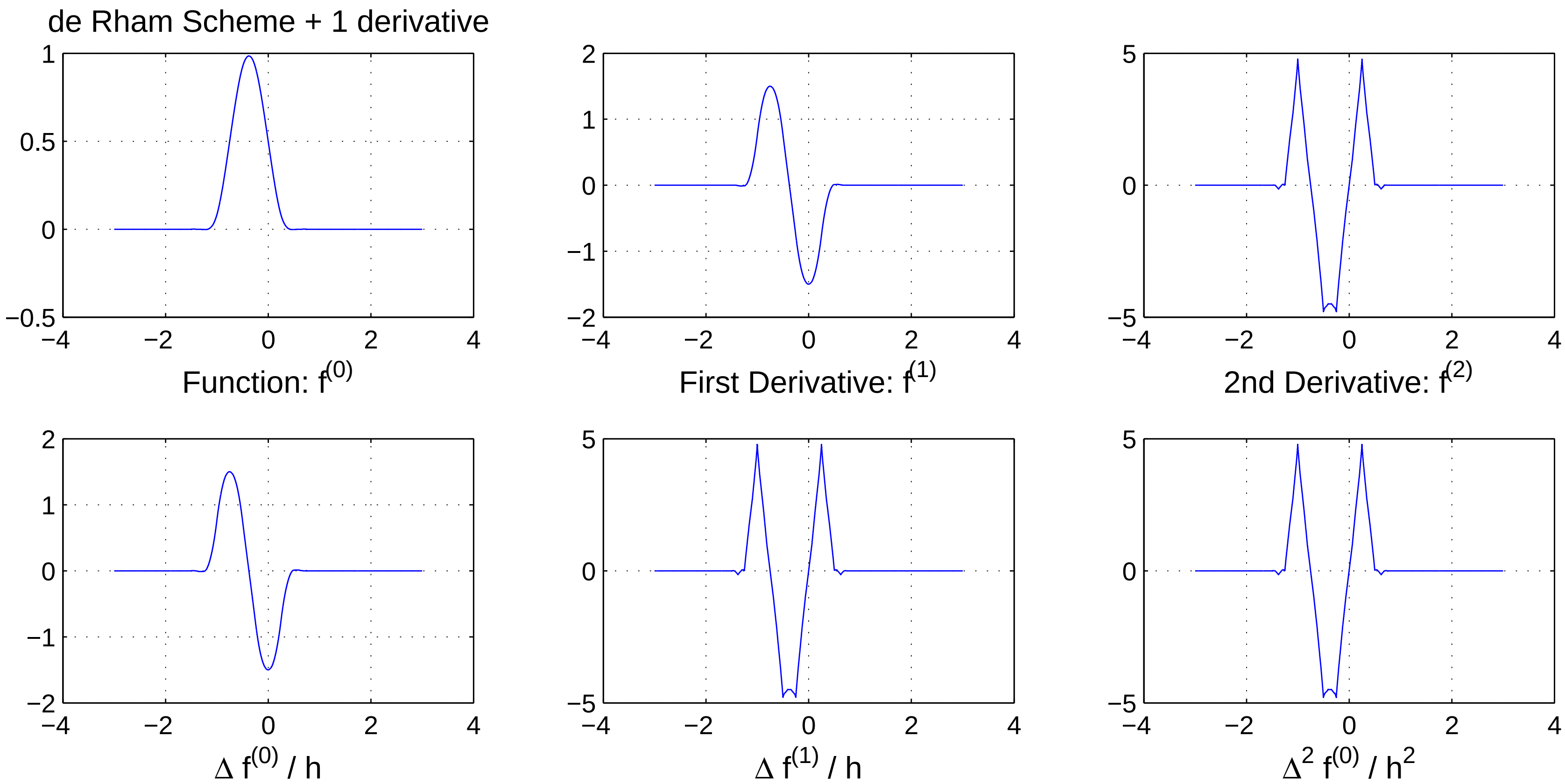
$$\lambda = -1/8,\ \bar{\mathcal{A}}_+^*(z) = \begin{bmatrix} \frac{(z+1)(\mu z^2+3z^2-2\mu z+10z+\mu+3)}{16z^2} & \cdots & \cdots & \frac{(z-1)(\mu z^2+2z^2+6z+\mu+2)}{32z^2} & 0 \\ \frac{(\mu-1)(\mu+2)(z-1)(z+1)^2}{4z^2} & \cdots & \cdots & \frac{(z+1)(\mu^2z^2+\mu z^2-z^2+2z+\mu^2+\mu-1)}{8z^2} & 0 \\ 0 & \cdots & \cdots & -\frac{(z-1)(z+1)^3}{8z^2} & 0 \end{bmatrix}$$

$$\widetilde{\bar{\mathcal{B}}_+}^*(z) = \begin{bmatrix} \frac{4\mu^2z^2+5\mu z^2-5z^2+4\mu^2z+2\mu z+2z+\mu+3}{4z} & \cdots & \cdots & -\frac{2z^3+4\mu^2z^2+5\mu z^2-4z^2+4\mu^2z+2\mu z+\mu+2}{8z} & \frac{z(2z+1)}{8} \\ \frac{(\mu-1)(\mu+2)(z-1)(z+1)}{z} & \cdots & \cdots & -\frac{(z-1)(z+1)(z+\mu^2+\mu-1)}{2z} & \frac{z(z+1)}{2} \\ 0 & \cdots & \cdots & -\frac{(z-1)(z+1)^2}{2z} & \frac{z(z+1)}{2} \end{bmatrix}$$

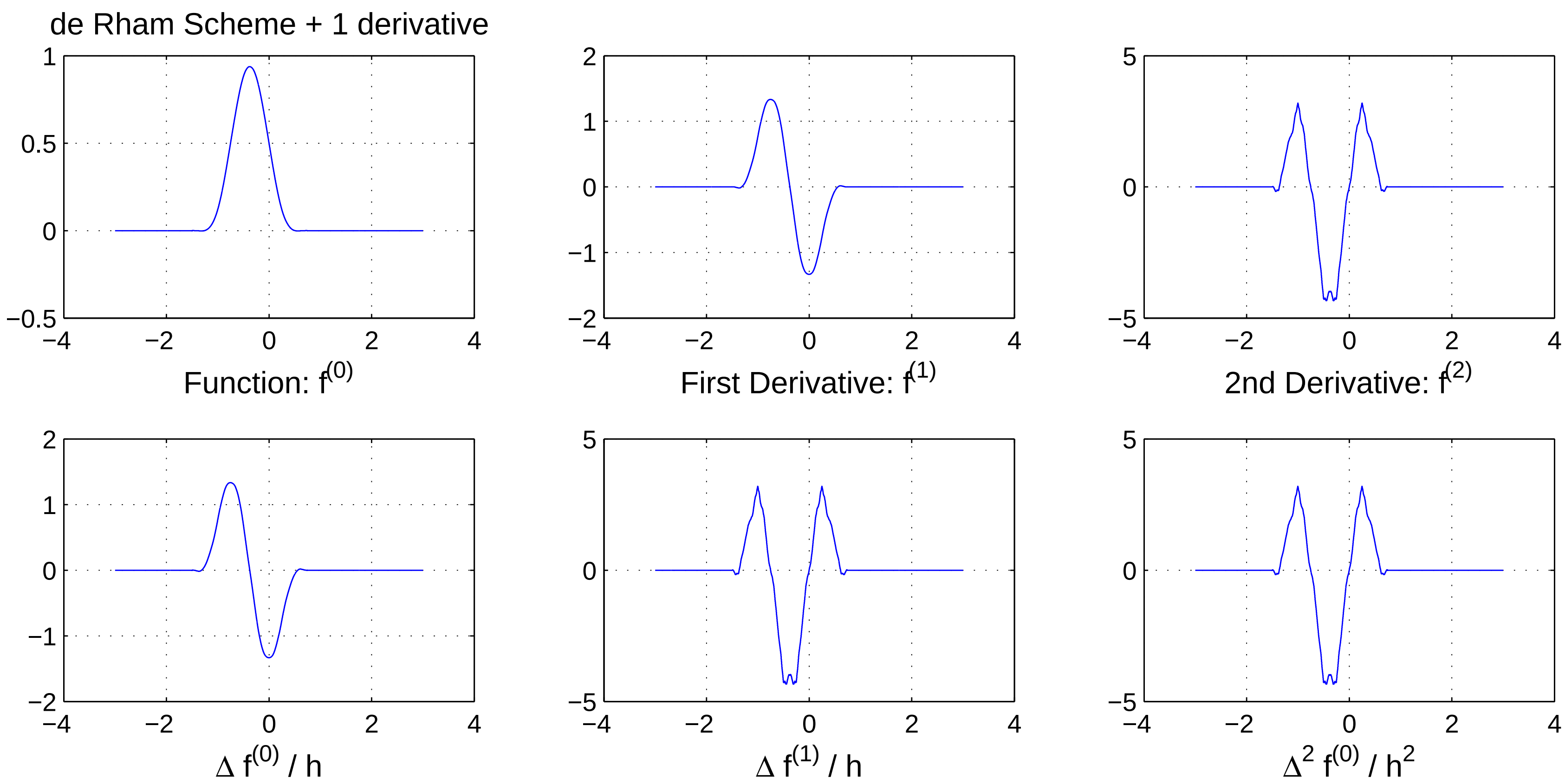
$\lambda = -1/8, \mu = -1$



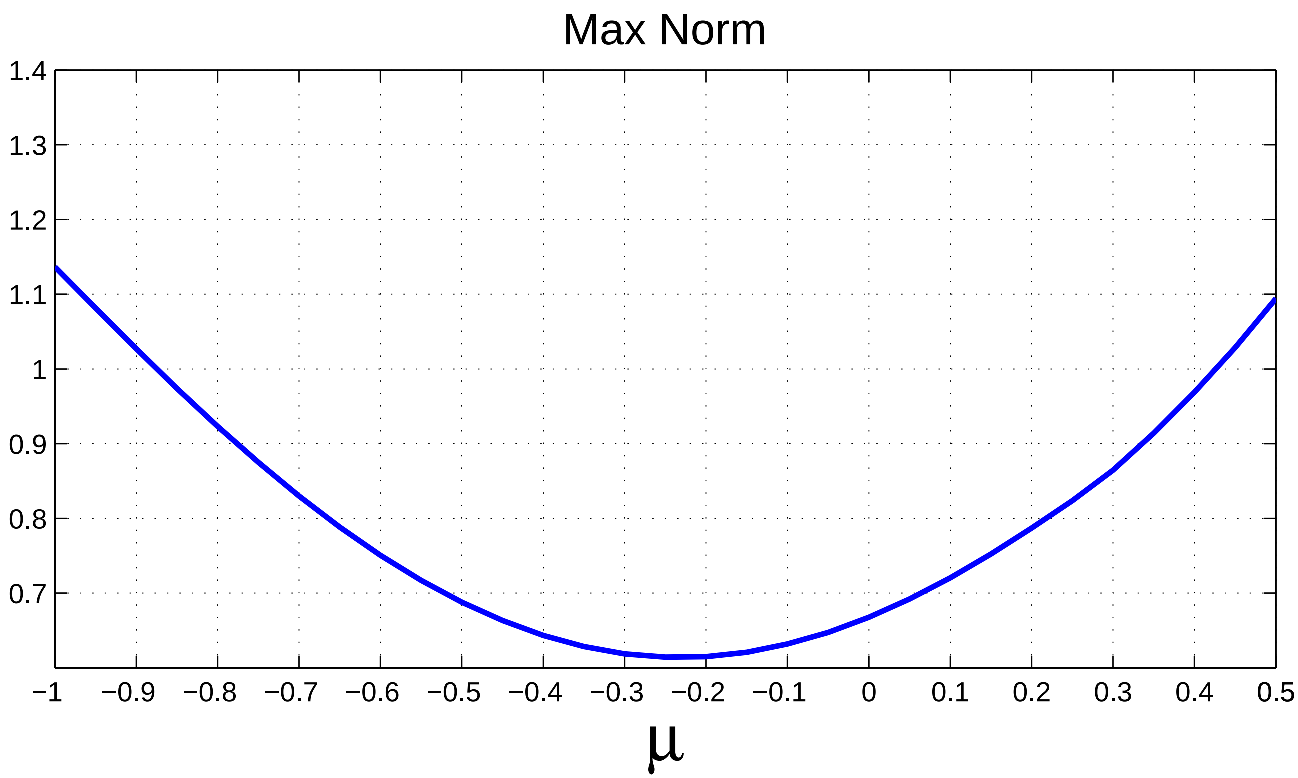
$\lambda = -1/8, \mu = -1/2$



$\lambda = -1/8, \mu = 0$



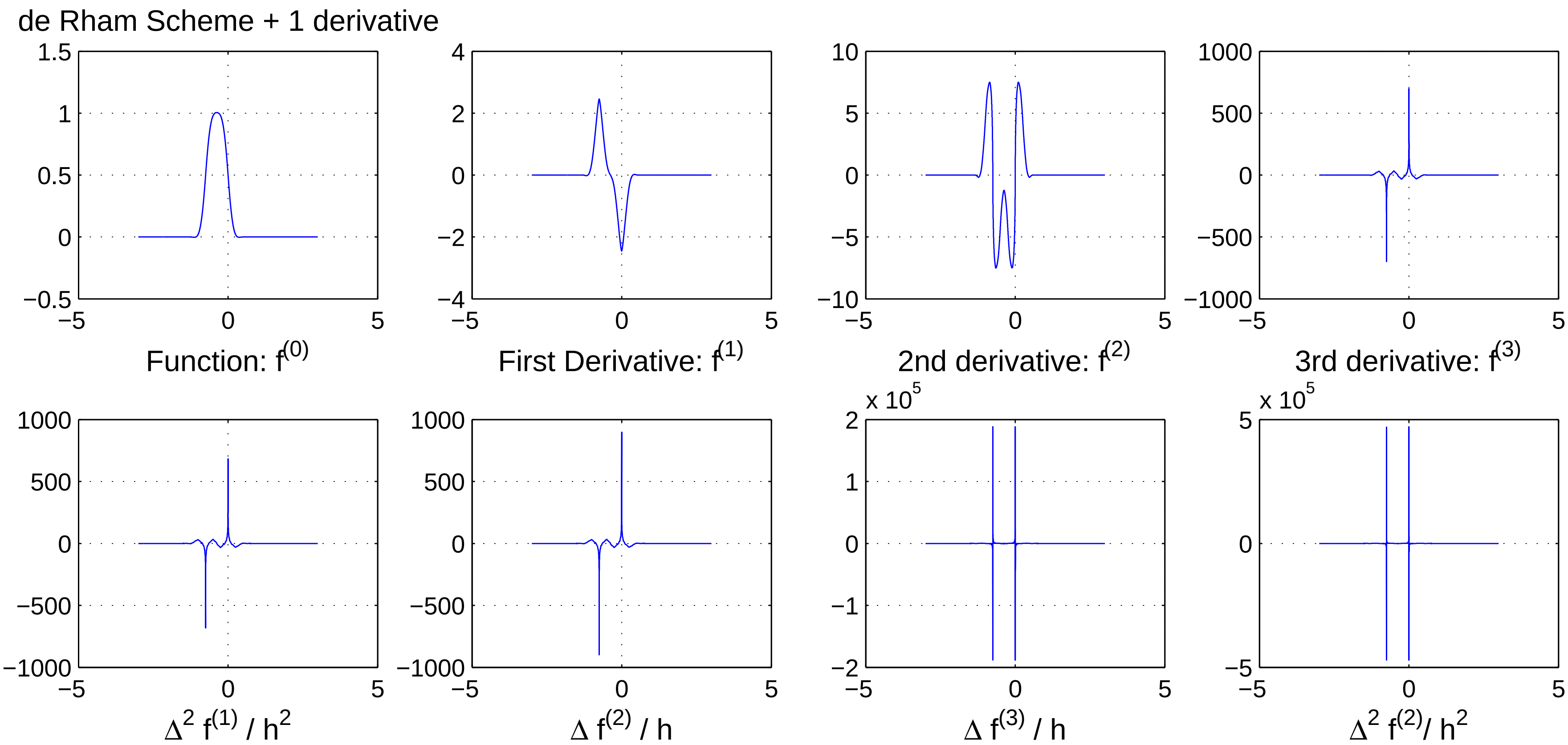
Contractivity of $S_{\widetilde{\bar{B}_+}}$



C^2-convergence of $H_{\bar{\boldsymbol{A}}}$ for $\mu \in [-0.85, 0.45]$.

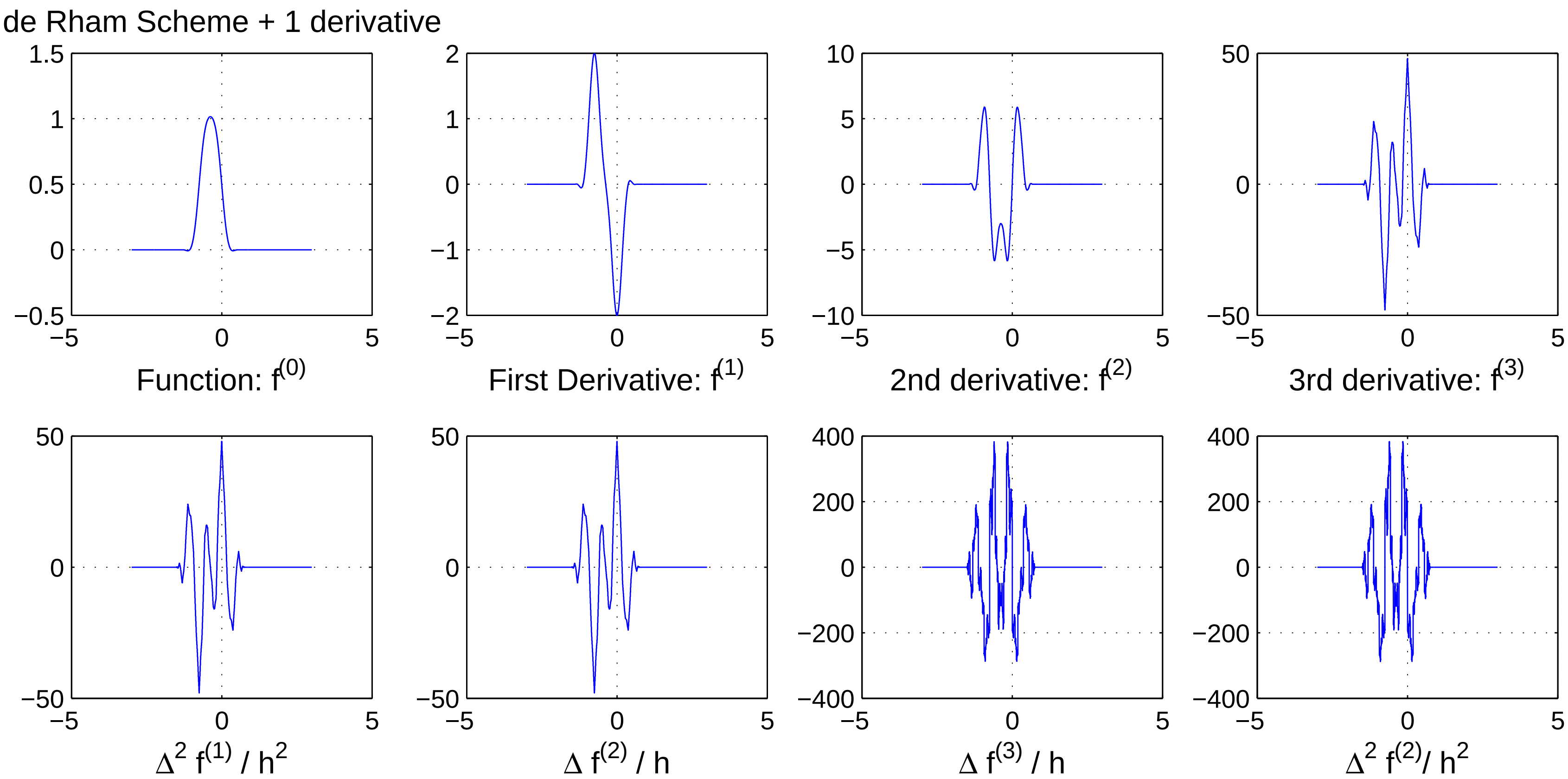
Extended De Rham from HC^2 Cubic

$$\alpha = -23/144; \beta = 9/4; \gamma = 3/2;$$



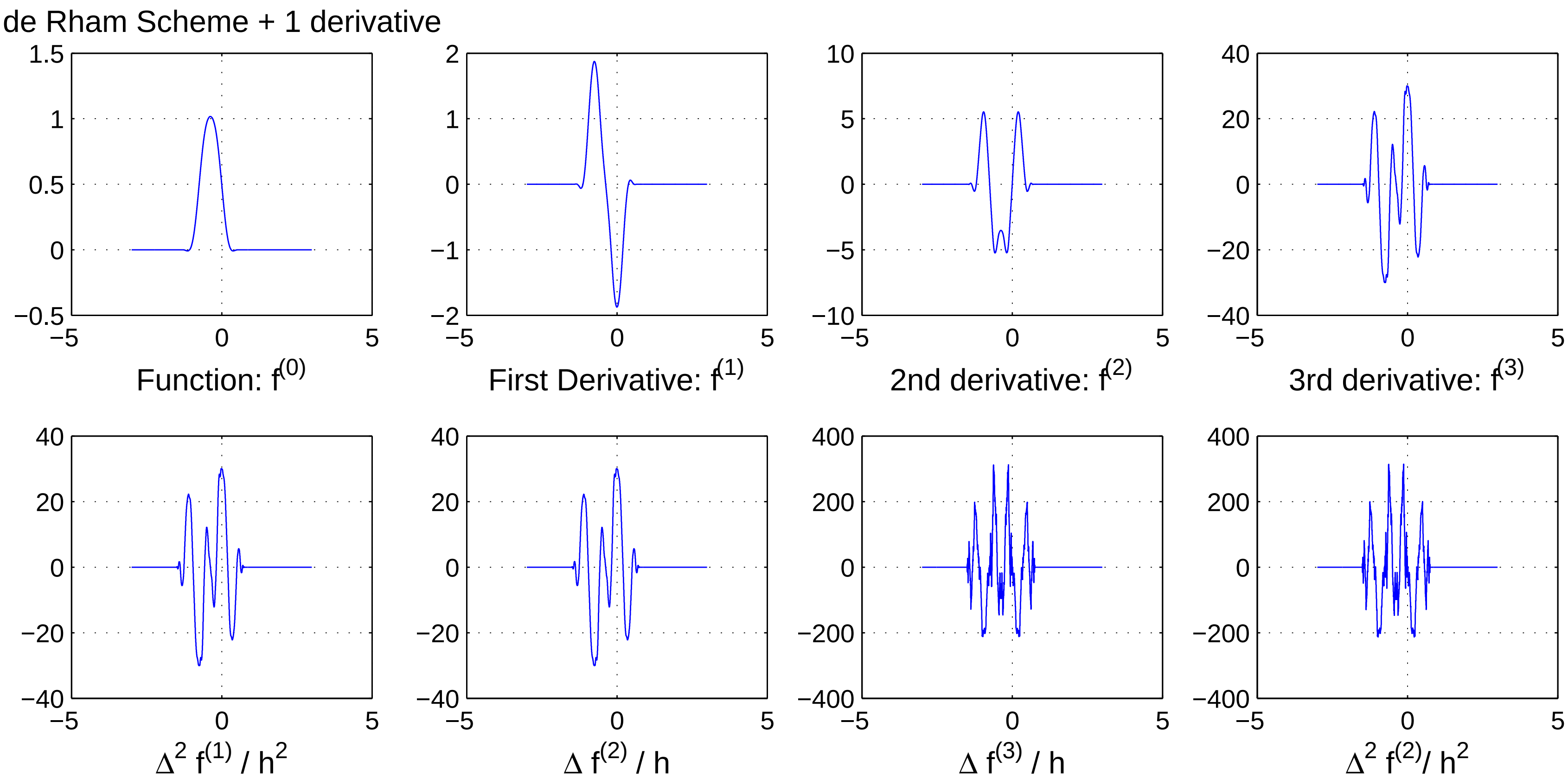
Extended De Rham from HC^2 Quartic

$$\alpha = -5/32; \beta = 2; \gamma = 3/2;$$



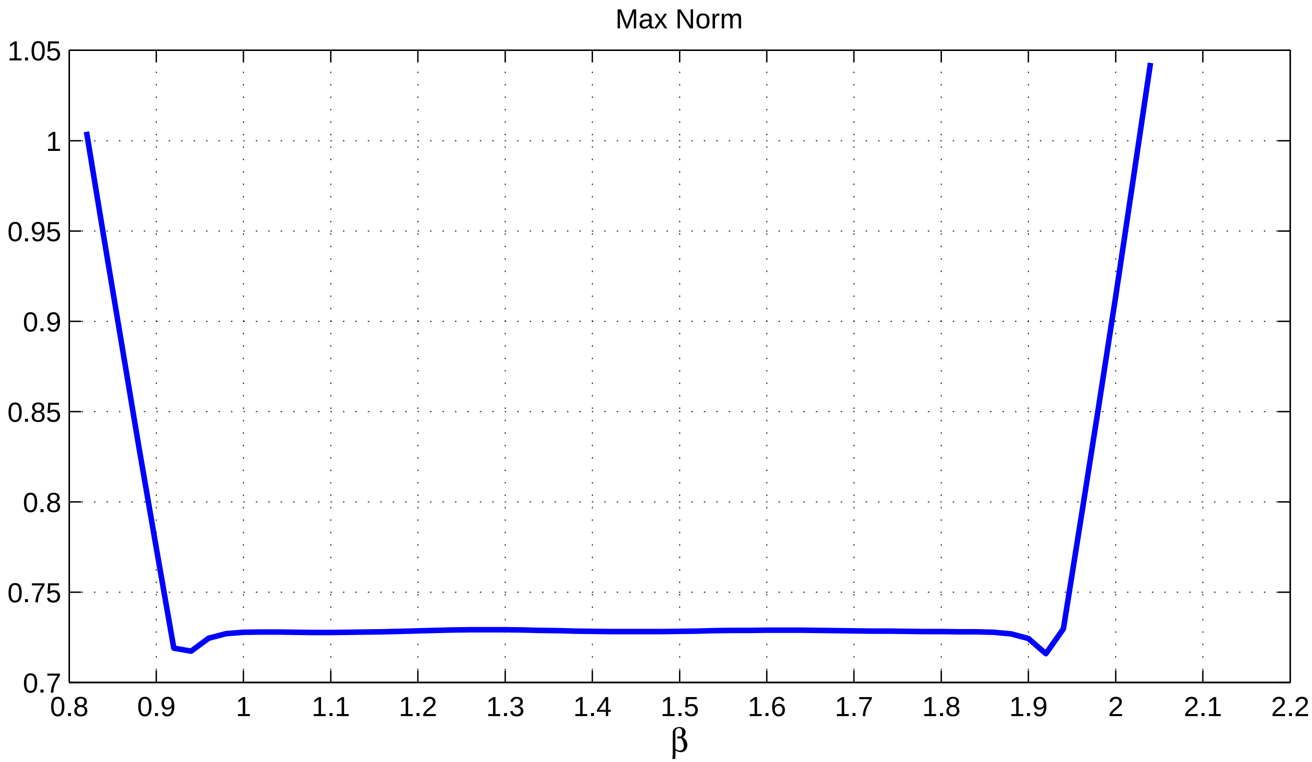
Extended De Rham from HC^2 Quintic

$$\alpha = -5/32; \beta = 15/8; \gamma = 3/2;$$



Contractivity of $S_{\widetilde{\bar{\boldsymbol{B}}_+}}$

$\alpha = -5/32; \gamma = 3/2;$



C^3-convergence of $H_{\bar{\boldsymbol{A}}}$ for $\beta \in [-0.82, 2.02]$.

6. The multivariable case, $s > 1$

The tools:

. Definition of Hermite scheme: OK

. Definition of spectral condition: OK

. Definition of Taylor operators: OK

. Theorem on factorizations with $S_{\boldsymbol{B}}$ and $S_{\tilde{\boldsymbol{B}}}$: OK

. Convergence: a little more difficult.

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. Definition of Taylor operators: OK

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. Convergence: a little more difficult.

The difficulty for any extension:

$$f_n^{(i+1,j)}(\alpha_1,\alpha_2) \simeq \frac{\partial^{i+1+j}\varphi\left(\frac{\alpha_1}{2^n},\frac{\alpha_2}{2^n}\right)}{\partial x_1^{i+1}\partial x_2^{j}} = \frac{\partial^{i+j+1}\varphi\left(\frac{\alpha_1}{2^n},\frac{\alpha_2}{2^n}\right)}{\partial x_1^{i}\partial x_2^{j}\partial x_1}$$

$\rightarrow$ The extension for a derivative in one direction has to be linked with the extensions in the other directions.