MULTIVARIATE APPROXIMATION AND INTERPOLATION WITH APPLICATIONS 2013

A class of Laplacian multi-wavelets bases for high-dimensional data

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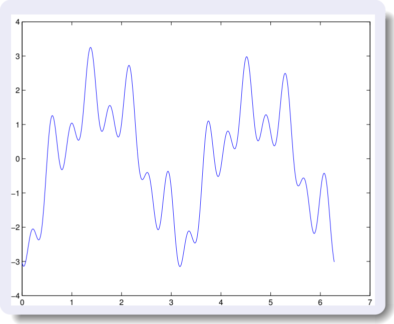
Joint work with Yoel Shkolnisky

A part of PhD thesis under the supervision of Yoel Shkolnisky and Nira Dyn

September 26, 2013

Representing signals



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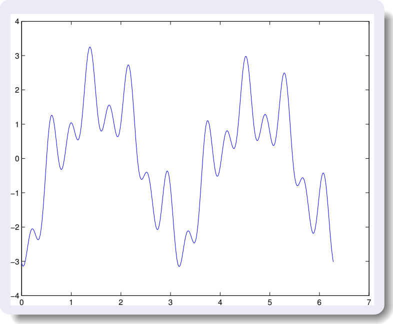

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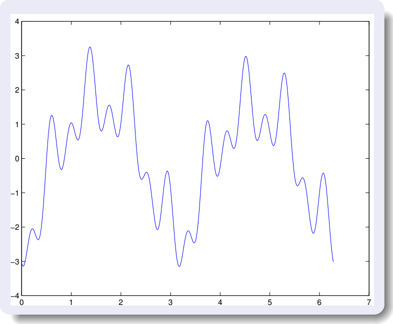

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- What to do in higher dimensions?
- What to do for general data - images, documents, gene arrays, . . . ?

What is general data?

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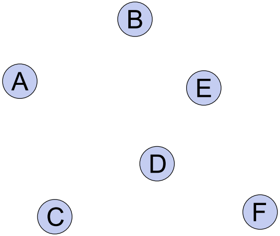

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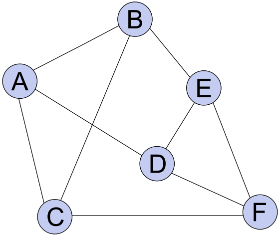

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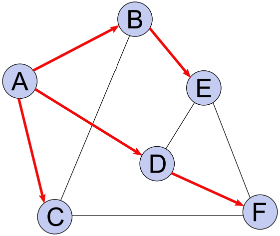
2. $\mathcal{X}$ has an associated tree structure – analog of a dydic partition.



The goal

Find N functions

$$\{\phi_n\}_{n=1}^N, \quad \phi_n : \mathcal{X} \mapsto \mathbb{R},$$

such that $\langle \phi_n, \phi_m \rangle = \delta_{n,m}$.

- We use

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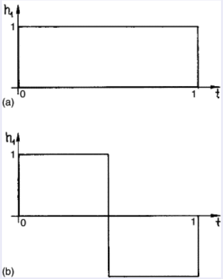
- Further requirements
  - The construction must be applicable in cases where D (the dimension of each point in $\mathcal{X}$) is very large.
  - It should allow for a sparse representation of a large family of functions.
  - It must have a fast and numerically stable algorithm.

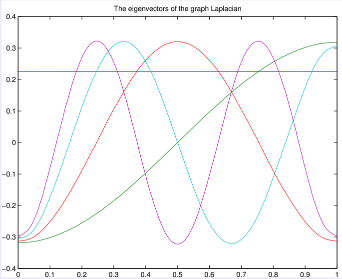
Known solutions

- Two known solutions for general data
  - Haar basis – Piecewise constant functions
  - Fourier basis – Eigenvectors of the (graph) Laplacian

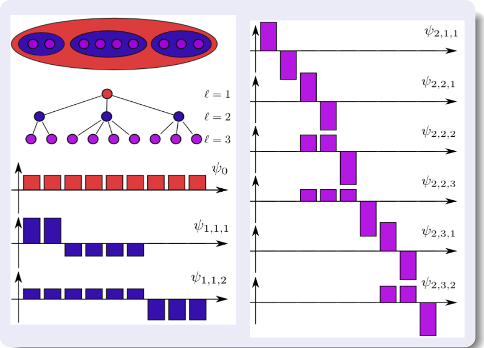
Haar basis



Fourier basis



Haar basis – general data



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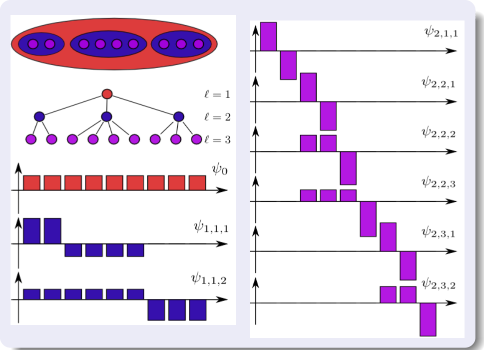

Haar-like on graphs (Gavish, Nadler, and Coifman)

Pros
- Simple, fast.
- Applicable to high dimensional data.

Cons
- Poor representations of smooth functions.

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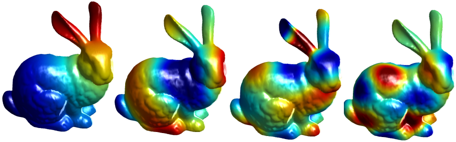
3. Compute the eigenvectors.

Graph Laplacian basis



Graph Laplacian's eigenvectors on meshes (Gabriel Peyré)

Pros
- Efficient representation for smooth functions.
- Applicable to high dimensional (almost arbitrary) data.

Cons
- Poor representation of non-smooth functions/rapidly changing functions.
- Global basis functions.

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- Building blocks: graph Laplacian and multi-resolution analysis.

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where $V_j = \mathbb{R}^N$, with N the number of data points.

2. Apply a fast orthogonalization process to obtain

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1},$$

with $W_p \perp V_p$, $W_p \oplus V_p = V_{p+1}$ for $p \geq 0$.

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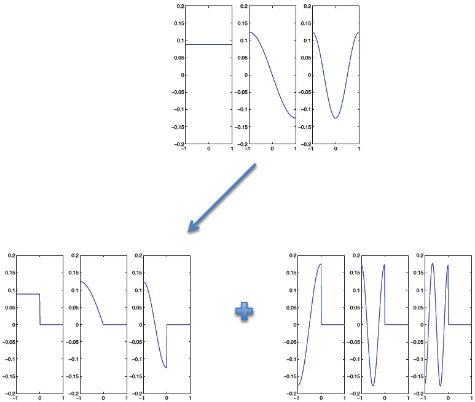
- This construction is repeated until V_j satisfies $\dim(V_j) = N$.

Approximation spaces – an example

Constructing $V_1 = V_{1,0} + V_{1,1}$



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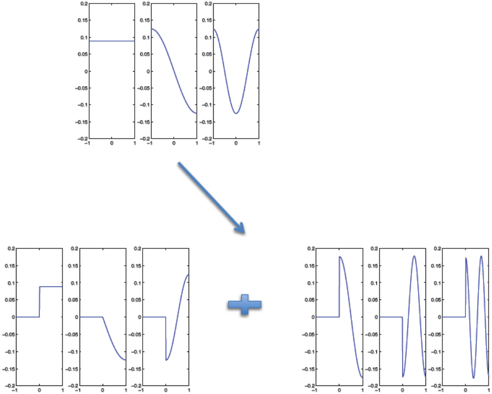

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3. We do not assume the tree is binary nor a complete tree.

Phase two

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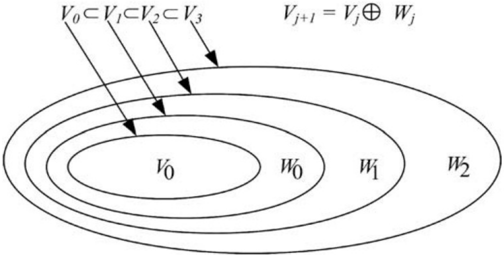
1. Recall that

$$V_j \subset V_J, \quad V_J \perp W_J \implies V_j \perp W_J, \quad 0 \le j < J.$$



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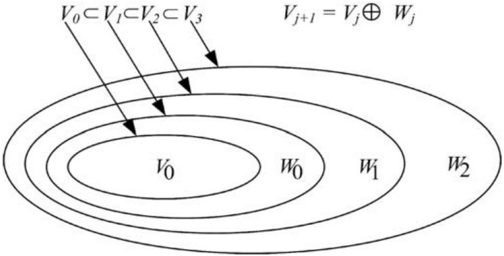

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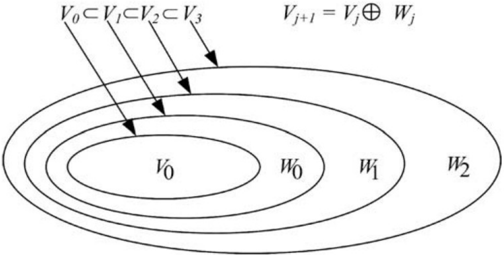


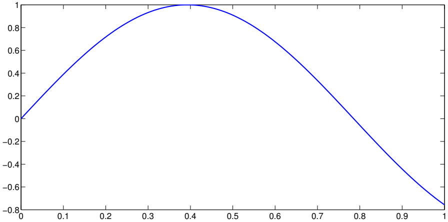
2. Due to sparsity, every complement space W_j is calculated with $\mathcal{O}(k^2 N)$ operations.
3. Overall complexity for this phase is $\mathcal{O}(k^2 N \log N)$. Usually $N \gg k$.

Representing functions (synthetic data)

The 1D case: taking 128 equally spaced on $[0,1]$. Compare the Haar ($k=1$), Laplacian ($k=N$), and an intermediate case ($1<k<N$)

Representing functions (synthetic data)

The function:



$\sin(4x)$

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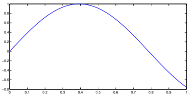

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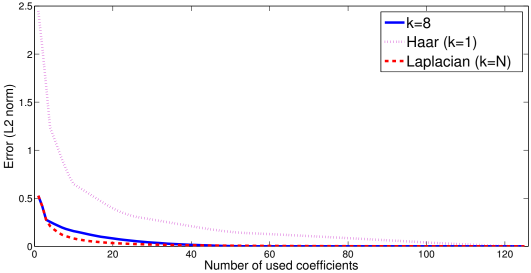

L_2 approximation error

Representing 1D functions – oscillatory function

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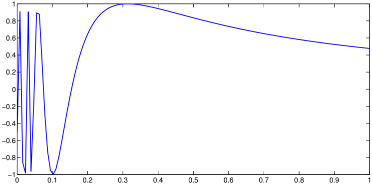

$$\sin\left(\frac{1}{0.01+2x}\right)$$

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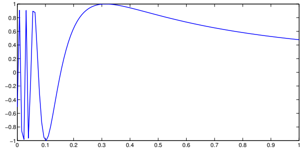

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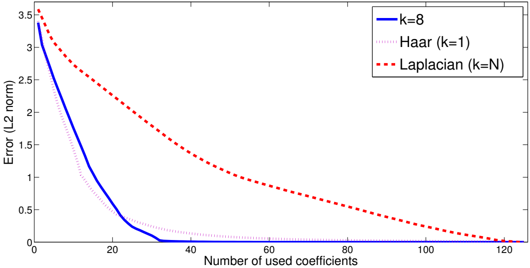
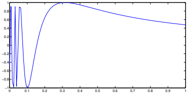


L_2 approximation error

Representing 1D functions – oscillatory function (cont.)



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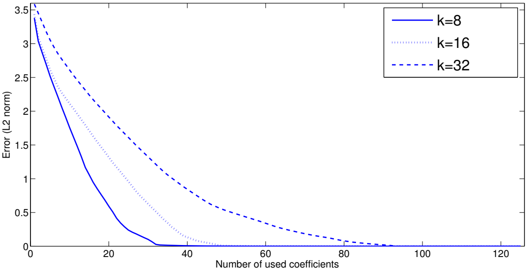
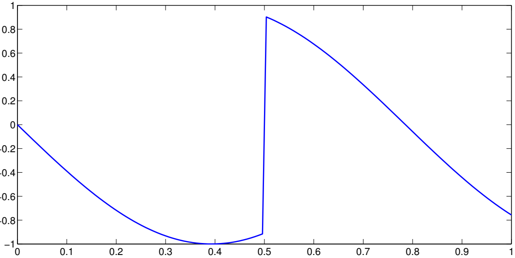


L_2 approximation error

Representing 1D functions – piecewise smooth function



$$\text{sign}(x - \tfrac{1}{2}) \sin(4x)$$

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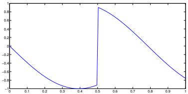

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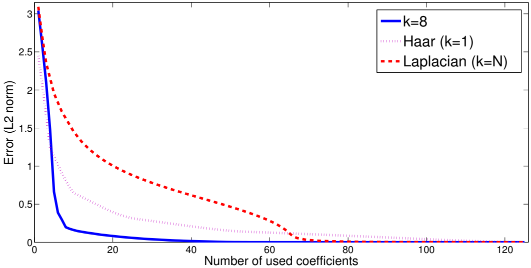
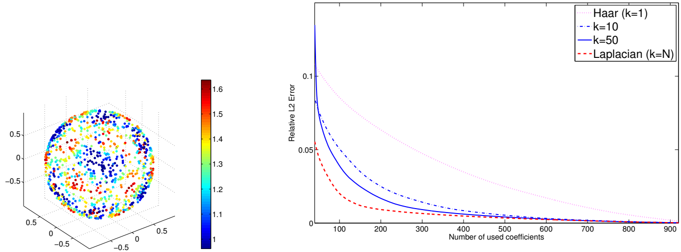


L_2 approximation error

A smooth function on $S^2 \subset \mathbb{R}^3$

A 3D case - 1000 data points distributed on the sphere. Compare between $k = 1, 10, 50, 1000$.

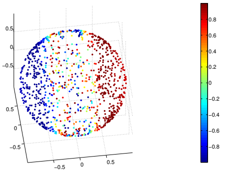
A smooth function on $S^2 \subset \mathbb{R}^3$



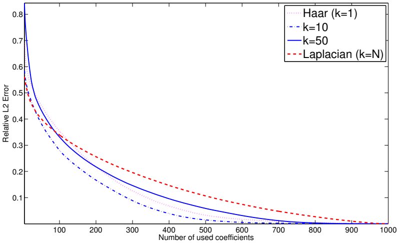
(a) $C(x) = \|\cos(2x)\|$ (b) L_2 relative error

Figure: Representing a smooth function on the sphere.

A rapidly changing function on $S^2 \subset \mathbb{R}^3$



(a) $R(x) = \sin\left((x^T x_0 + 0.2)^{-1}\right)$



(b) L_2 relative error

Figure: R oscillates rapidly in regions on the sphere where x close to be orthogonal to x_0.

Compression of hyper spectral images

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- In this example, hyperspectral image of visible spectral region:

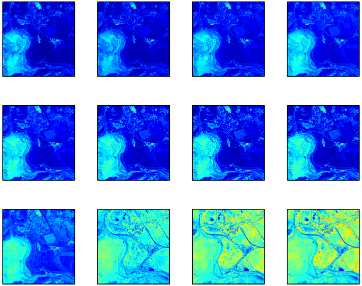

Figure: The 12 different wave length images given as the data.

Compression of surface temperature

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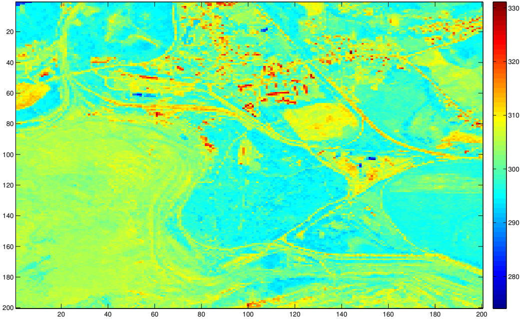
- The surface temperature is derived form sensors in non-visible spectral region - long-wave infrared sensors. Measured in Kelvin.



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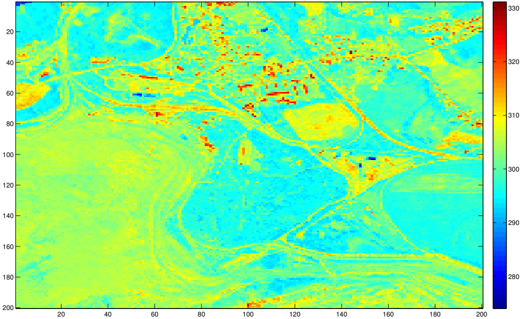
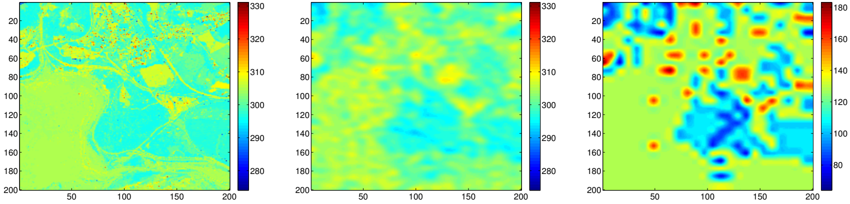


- We compare the compression with two (non-adaptive) benchmarks: DCT and JPEG2000.

Compression results

Using 200 coefficients, that is 0.5%:



(a) LMW (b) DCT (c) JPEG2000

Thank you !

Questions ?