Convergence of uniform subdivision

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Generalizations: $p < \infty$, vector subdivision, fast convergence, infinite mask, non-dyadic dilations...

mask and dilation

$a \in \mathbb{C}^{\mathbb{Z}^d/2}$ is a finite mask. Considered as a discrete finite measure

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Definition: the cascade operator C

$$C : f \mapsto \mathcal{D}f * a.$$

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Question

Given a compactly supported g, do we have

$$\|C^k g - \phi\|_\infty \to 0?$$

Each of the following conditions is necessary:

- $g, \phi \in C^\alpha$, $\alpha \geq 0$.
- $\sum_{j \in \gamma + 2\mathbb{Z}^d} a(j) = 1$, $\gamma \in \{0, 1\}^d$.
- $g - \phi$ has zero mean.
- The PSI space $S(g)$ provides approximation order 1 in the ∞-norm, viz., for each sufficiently smooth f, as $k \to \infty$,

$$\mathrm{dist}_{L_\infty}(f, \mathcal{D}^k S(g)) = O(2^{-k}).$$

$$G_0$$

is the collection of compact support g that satisfy the above.

Subdivision: definition and convergence

Definition: the space $\mathcal{Q}_k$

$\mathcal{Q}_k := \mathbb{C}^{\mathbb{Z}^d/2^k}$

Definition: The subdivision operator S_k, convergence

$$S_k : \mathcal{Q}_0 \to \mathcal{Q}_k, \quad \lambda \mapsto \mathcal{D}^{k-1} a * S_{k-1}\lambda.$$

Convergence:

$$\|\mathcal{D}^k g * S_k\delta - \phi\|_\infty \to 0, \quad \forall g \in G_0.$$

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$C^k g = \mathcal{D}^k g * S_k\delta$

$$\sum_{j=1}^{k} \mathcal{D}^{j-1} + \mathcal{D}^k = 1 + \mathcal{D} \sum_{j=1}^{k} \mathcal{D}^{j-1}.$$

The L_2-case is spectral

The Transfer operator T

With f a trig. pol., and $\tau := |\widehat{a}|^2$,

$$T : f \mapsto \mathcal{D}^{-1}\Big(\sum_{\gamma \in \{0,1\}^d} (\tau f)(\cdot + \pi\gamma)\Big).$$

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The transfer operator encodes L_2-properties of a and ϕ, including a complete characterization of the convergence of cascade: essentially it need to have a unique dominant eigenvalue (acting on any large enough set of trig. pol.).

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The transfer cannot be used (obvious reasons) for other norms.

Characterization: joint spectral radius

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Despite of its name, the joint spectral radius is joint but not spectral.

Convergence depends on dependence relations

The space K_ϕ

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Convergence of cascade: (more or less) we need that

$$S_k(K_\phi) \to 0.$$

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If $\dim K_\phi < \infty$, then spectral.

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Special case: box splines, de Boor-R

If ϕ is a box spline, then spectral.