Interpolation problems on cycloidal spaces

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$$P_n := \langle 1, x, \ldots, x^n \rangle$$

Hermite interpolation problems on P_n

For $x_0, \ldots, x_n$, not necessarily distinct, find $p \in P_n$ such that

$$\lambda_i p = \lambda_i f, \quad i = 0, \ldots, n,$$

where

$$\lambda_i f := f^{(r_i - 1)}(x_i), \quad r_i = \#\{j \leq i \mid x_j = x_i\}.$$

The Hermite interpolation problem in P_n has always a unique solution $P(f; x_0, \ldots, x_n)$ for any set of nodes $x_0, \ldots, x_n$.

Newton basis functions

Given $x_0, \ldots, x_{n-1}$, not necessarily distinct, the **Newton basis function**

$$\omega_n(x) := (x - x_0) \cdots (x - x_{n-1}).$$

For $n = 0$, define $\omega_0(x) = 1$.

The Newton basis function $\omega_n(x)$ is a function in P_n vanishing on $x_0, \ldots, x_{n-1}$ and whose coefficient in x^n with respect to the basis $(1, x, \ldots, x^n)$ is 1.
This function can be regarded as the interpolation error of the function x^n at $x_0, \ldots, x_{n-1}$

$$\omega_n(x) = x^n - P\big((\cdot)^n; x_0, \ldots, x_{n-1}\big)(x)$$

The set of functions $(\omega_k(x))_{k=0,\ldots,n}$ form a basis of P_n.

Divided difference

$[x_0, \dots, x_n]f$ is the coefficient in x^n with respect to the basis $(1, x, \dots, x^n)$ of the interpolant $P(f; x_0, \dots, x_n)$

Newton interpolation formula

$$P(f; x_0, \dots, x_n)(x) = \sum_{k=0}^{n} [x_0, \dots, x_k]f\, \omega_k(x)$$

Aitken-Neville formula and recurrence relations

Neville formula for the polynomial interpolant

$$P(f;x_0,\dots,x_n) = \frac{x_n - x}{x_n - x_0} P(f;x_0,\dots,x_{n-1}) + \frac{x - x_0}{x_n - x_0} P(f;x_1,\dots,x_n).$$

Recurrence relations for divided differences

$$[x_0,\dots,x_n]f = \frac{[x_1,\dots,x_n]f - [x_0,\dots,x_{n-1}]f}{x_n - x_0}.$$

Divided differences at a single point

$$[x_0,\dots,x_n]f = \frac{f^{(n)}(x_0)}{n!}, \quad x_0 = \cdots = x_n.$$

Follows from Taylor formula: $P(f;x_0,\dots,x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$.

Given $u_0, \ldots, u_n$ a set of linearly independent functions,

$$U_n = \langle u_0, \ldots, u_n \rangle$$

Hermite interpolation problems on U_n:

For $x_0, \ldots, x_n$, not necessarily distinct, find $u \in U_n$ such that

$$\lambda_i u = \lambda_i f, \quad i = 0, \ldots, n,$$

where $\lambda_i f := f^{(r_i - 1)}(x_i)$, $r_i = \#\{j \le i \mid x_j = x_i\}$.

Hermite interpolation problems do not have a solution for any set of nodes. A condition is required.

Extended collocation matrices

If we express the interpolant $u(x) = \sum_{k=0}^{n} c_k u_k(x)$ in terms of the given basis, the interpolation conditions lead to a linear system

$$\mathcal{M}^*\begin{pmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \lambda_0 f \\ \vdots \\ \lambda_n f \end{pmatrix}$$

whose coefficient matrix

$$\mathcal{M}^*\begin{pmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{pmatrix} := (u_j^{(m_i)}(t_i))_{0\le i\le n;\, 0\le j\le n}, \; m_i := \#\{k < i | x_k = x_i\}$$

is called the *extended collocation matrix*.

Theorem

The Hermite problem has a unique solution if and only if the corresponding extended collocation matrix has nonzero determinant.

Definition

A system of functions is extended Chebyshev (ET) on $[a, b]$ if all extended collocation matrices have positive determinants,

$$\det \mathcal{M}^* \begin{pmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{pmatrix} > 0, \text{ for all } x_0 \leq \cdots \leq x_n \text{ in } [a, b].$$

An extended Chebyshev (ET) space is a space generated by an extended Chebyshev basis.

If U_n is ET, then the Hermite interpolation problem at an arbitrary extended sequence of nodes has always a unique solution.

In order to derive Newton and Aitken-Neville formulae, it is required that interpolation problems at $k \leq n$ nodes $x_i, \dots, x_{i+k}$ have a unique solution in the space $U_k = \langle u_0, \dots, u_k \rangle$.

Definition

A system of functions is extended complete Chebyshev (ECT) on $[a, b]$ if all systems $(u_0, \dots, u_k)$, $k = 0, \dots, n$, are extended Chebyshev.

Mühlbach derived Newton formulae and Aitken-Neville formulae for ECT spaces on $[a, b]$.

Remark

An ET space on $[a, b]$ is ECT on sufficiently small subintervals. The hypothesis that we have a ECT basis might lead to strong restrictions on the domain. For our purposes we need to discuss the validity of the formulae under weaker hypotheses.

Critical length

J. M. Carnicer, E. Mainar, J. M. Peña; *Critical Length for Design Purposes and Extended Chebyshev Spaces,* Const. Approx. **20**, 55–71.

Definition

Let U_n be a space of differentiable functions which is invariant under translations. The *critical length* of U is the number $\ell_n \in (0, +\infty]$ such that U is ET on any interval I if and only if I does not contain a compact interval of length ℓ_n.

Proposition

Let U_n be an $(n+1)$-dimensional space of differentiable functions which is invariant under translations and reflections. Let $(u_0, \dots, u_n)$ be a basis such that $W(u_0, \dots, u_n)(0)$ is lower triangular with positive diagonal entries. Then U_n is an ET space on each interval of length less than or equal to α if and only if

$$w_{k,n}(x) := \det W(u_k, \dots, u_n)(x) > 0, \quad \forall k > n/2, \quad t \in (0, \alpha].$$

If the space is invariant under reflections, the critical length can be identified as the first positive zero of the functions $w_{k,n}$, $k > n/2$, that is

$$\ell_n := \min_{k>n/2} \min\{\alpha; w_{k,n}(\alpha) = 0, \alpha > 0\}.$$

General Cycloidal spaces C_n

$$C_1 := \langle \cos x, \sin x \rangle$$
$$C_n := \Big\langle \cos x, \sin x, 1, x, \dots, x^{n-2} \Big\rangle, \quad n \geq 2$$

An alternative basis to $(\cos x, \sin x, 1, x, x^2, \dots, x^{n-2})$ for C_n is given by

$$\varphi_0(x) := \cos x,$$
$$\varphi_i(x) := \int_0^x \varphi_{i-1}(y)dy = \frac{1}{(i-1)!}\int_0^x (x-t)^{i-1} \cos t dt, \;\; i = 1, \dots, n.$$

Clearly, $\varphi_k \in C_k$ and

$$\varphi_k(0) = \varphi_k'(0) = \cdots = \varphi_k^{(k-1)}(0) = 0, \quad \varphi_k^{(k)}(0) = 1.$$

Fundamental functions

$$\varphi_0(x) = \cos x = 1 - \frac{x^2}{2!} + \cdots ,$$
$$\varphi_1(x) = \sin x = x - \frac{x^3}{3!} + \cdots ,$$
$$\varphi_2(x) = 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots ,$$
$$\varphi_3(x) = x - \sin x = \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots ,$$
$$\varphi_4(x) = \cos x - 1 + \frac{x^2}{2!} = \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$
$$\varphi_5(x) = \sin x - x + \frac{x^3}{3!} = \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

Hermite interpolation problems on C_n

For $x_0, \ldots, x_n$, not necessarily distinct, find $c \in C_n$ such that

$$\lambda_i c = \lambda_i f, \quad i = 0, \ldots, n, \quad \lambda_i f := f^{(r_i - 1)}(x_i), \quad r_i = \#\{j \leq i \mid x_j = x_i\}.$$

Hermite interpolation problems on cycloidal spaces do not have solution on any sequence of nodes.

Notation

If the solution exists and is unique, we denote it by $C(f; x_0, \ldots, x_n)$.

Existence of the cycloidal interpolant

They are spaces invariant under translations and reflections and therefore they have a critical length ℓ_n.

Proposition

The cycloidal space C_n is ET on $[a, b]$ if $b - a < \ell_n$

Therefore a sufficient condition on $x_0 \leq \cdots \leq x_n$ for the existence of solution of the Hermite interpolation problem is that $x_n - x_0 < \ell_n$.

A Taylor formula

The Taylor problem has solution because $x_0 = \cdots = x_n$ and $x_n - x_0 = 0 < \ell_n$.

Taylor formula

$$C(f; x_0^{[n+1]})(x) = \sum_{k=0}^{n-2} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + \sum_{k=n-1}^{n} f^{(k)}(x_0)\varphi_k(x-x_0).$$

Critical length of cycloidal spaces

J. M. Carnicer, E. Mainar, J. M. Peña, *On the critical lengths of cycloidal spaces*, to appear in Constructive Approximation

Theorem

The critical length ℓ_n of the cycloidal spaces C_n is non-decreasing and satisfies $\ell_{2k} = \ell_{2k+1} < \ell_{2k+2}$ for each $k \geq 1$. The critical length $\ell_{2k} = \ell_{2k+1}$ is the first positive zero of $w_{k+1,2k} = \det W(\varphi_{k+1}, \dots, \varphi_{2k})$

The above theorem permits the computation of the critical lengths

Critical lengths

The critical length of the cycloidal spaces is non-decreasing

$$\ell_2 = \ell_3 = 2\pi \approx 6.28318 < \ell_4 = \ell_5 \approx 8.9868 < \ell_6 = \ell_7 \approx 11.5269$$

Proposition

The cycloidal space C_n is ECT on $[a, b]$ if $b - a < \ell_1 = \pi$.

Newton formulae and Aitken-Neville formulae will work on intervals of length less than π.

However we can extend this formulae to a wider class of interpolation problems:

—Newton Formula holds, if $x_k - x_0 < \ell_k$, $k = 2, \dots, n$.

—Neville Formula holds, if $max_{i=0,\dots,n-k}(x_{i+k} - x_i) < \ell_k$, $k = 2, \dots, n$.

Cycloidal Newton basis and cycloidal divided differences

Cycloidal Newton basis functions

Given $x_0, \ldots, x_{n-1}$ ($n \geq 2$), not necessarily distinct such that the Hermite interpolation problem has a unique solution in C_{n-1}, we define the *cycloidal Newton basis function*

$$\omega(x; x_0, \ldots, x_{n-1}) := x^{n-2} - C\big((\cdot)^{n-2}; x_0, \ldots, x_{n-1}\big)(x)$$

Cycloidal divided differences

If the Hermite interpolation problem at $x_0, \ldots, x_n$, $n \geq 2$ has a unique solution, we define the *cycloidal divided difference* $[x_0, \ldots, x_n]_C f$ as the coefficient in x^{n-2} with respect to the basis $(\cos x, \sin x, 1, x, \ldots, x^{n-2})$ of the cycloidal interpolant $C(f; x_0, \ldots, x_n)$.

Cycloidal Newton interpolation formula

Let $x_0, \dots, x_n$, $n \geq 2$, be an extended sequence such that the Hermite interpolation problems on C_k at $x_0, \dots, x_k$, for $k = 1, \dots, n$, have a unique solution. Then we have

$$C(f; x_0, \dots, x_n)(x) = C(f; x_0, x_1)(x) + \sum_{k=2}^{n} [x_0, \dots, x_k]_C f \, \omega(x; x_0, \dots, x_{k-1})$$

Remark: The hypotheses hold if $x_k - x_0 < \ell_k$ for any $k \geq 1$.

Aitken-Neville formula for the cycloidal interpolant

Let $x_0, \dots, x_n$, $n \geq 3$, be an extended sequence such that the Hermite interpolation problems on the corresponding cycloidal spaces have a unique solution for the sequences of nodes $(x_0, \dots, x_n)$, $(x_0, \dots, x_{n-1})$, $(x_1, \dots, x_n)$ and $(x_1, \dots, x_{n-1})$. Then we have

$$\left([x_0, \dots, x_{n-1}]_C(\cdot)^{n-2} - [x_1, \dots, x_n]_C(\cdot)^{n-2}\right) C(f; x_0, \dots, x_n) =$$
$$\frac{\omega(x; x_1, \dots, x_n)}{\omega(x; x_1, \dots, x_{n-1})} C(f; x_0, \dots, x_{n-1}) - \frac{\omega(x; x_0, \dots, x_{n-1})}{\omega(x; x_1, \dots, x_{n-1})} C(f; x_1, \dots, x_n).$$

Recurrence relations for cycloidal divided differences

Let $x_0, \ldots, x_n$, $n \geq 3$, be an extended sequence such that the Hermite interpolation problems on the corresponding cycloidal spaces have a unique solution for the sequences of nodes

$$(x_0, \ldots, x_n), \quad (x_1, \ldots, x_n), \quad (x_0, \ldots, x_{n-1}), \quad (x_1, \ldots, x_{n-1}).$$

Then we have

$$[x_0, \ldots, x_n]_C f = \frac{[x_1, \ldots, x_n]_C f - [x_0, \ldots, x_{n-1}]_C f}{[x_0, \ldots, x_{n-1}]_C (\cdot)^{n-2} - [x_1, \ldots, x_n]_C (\cdot)^{n-2}}.$$

Cycloidal divided differences at a single point

$$[x_0, \ldots, x_n]_C f = \frac{f^{(n-2)}(x_0) + f^{(n)}(x_0)}{(n-2)!}, \quad x_0 = \cdots = x_n.$$

Efficient computation of cycloidal divided differences

Notation: $d_{i,k}f := [x_i, \ldots, x_{i+k}]_C f$.

$$d_{i,k}(\cdot)^j = \begin{cases} \dfrac{d_{i+1,k-1}(\cdot)^j - d_{i,k-1}(\cdot)^j}{d_{i+1,k-1}(\cdot)^{k-2} - d_{i,k-1}(\cdot)^{k-2}}, & \text{if } x_i < x_{i+k}, \\ x_i^{j-k}\Big(\binom{j}{k-2}x_i^2 + k(k-1)\binom{j}{k}\Big), & \text{otherwise,} \end{cases}$$

and

$$d_{i,k}f = \begin{cases} \dfrac{d_{i+1,k-1}f - d_{i,k-1}f}{d_{i+1,k-1}(\cdot)^{k-2} - d_{i,k-1}(\cdot)^{k-2}}, & \text{if } x_i < x_{i+k}, \\ \dfrac{f^{(k-2)}(x_i) + f^{(k)}(x_i)}{(k-2)!}, & \text{otherwise.} \end{cases}$$

Efficient computation of Newton's functions

$$\omega(x; x_0, \dots, x_{j-1}) = \omega(x; x_0, x_1) - \sum_{k=2}^{j-1} d_{0,k}(\cdot)^{j-2}\,\omega(x; x_0, \dots, x_{k-1}).$$

with

$$\omega(x; x_0, x_1) = \begin{cases} \dfrac{\sin\frac{x-x_0}{2}\sin\frac{x-x_1}{2}}{\sin\frac{x_1-x_0}{2}}, & \text{if } x_1 \neq x_0, \\ 2\sin^2\dfrac{x-x_0}{2}, & \text{if } x_1 = x_0, \end{cases}$$

Finally, the cycloidal interpolant is given by

$$C(f; x_0, \dots, x_n) = C(f; x_0, x_1) + \sum_{k=2}^{n} d_{0,k} f\,\omega(x; x_0, \dots, x_{k-1}).$$

Relation between cycloidal and polynomial interpolants

Theorem

The Hermite interpolation problem at the extended sequence $x_0, \ldots, x_n$, $n \geq 1$, has a unique solution in C_n if and only if

$$[x_0, \ldots, x_{n-1}] \cos \cdot [x_0, \ldots, x_n] \sin \neq [x_0, \ldots, x_{n-1}] \sin \cdot [x_0, \ldots, x_n] \cos .$$

$$C(f; x_0, \ldots, x_n)(x) = P(f; x_0, \ldots, x_n)(x) + a_0 e_0(x) + a_1 e_1(x),$$

where $e_0(x) := \cos(x) - P(\cos; x_0, \ldots, x_n)(x)$,
$e_1(x) := \sin(x) - P(\sin; x_0, \ldots, x_n)(x)$, and (a_0, a_1) is the solution of the linear system

$$\begin{pmatrix} [x_0, \ldots, x_{n-1}] \cos & [x_0, \ldots, x_{n-1}] \sin \\ [x_0, \ldots, x_n] \cos & [x_0, \ldots, x_n] \sin \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} [x_0, \ldots, x_{n-1}] f \\ [x_0, \ldots, x_n] f \end{pmatrix} .$$

Theorem

Let $x_0, \ldots, x_n$, $n \geq 2$, be an extended sequence such that there exists a unique solution to the corresponding Hermite interpolation problem. Then

$$[x_0, \ldots, x_n]_C f = [x_0, \ldots, x_{n-2}] f - a_0 [x_0, \ldots, x_{n-2}] \cos - a_1 [x_0, \ldots, x_{n-2}] \sin,$$

where a_0, a_1 are the solutions of the linear system

$$\begin{pmatrix} [x_0, \ldots, x_{n-1}] \cos & [x_0, \ldots, x_{n-1}] \sin \\ [x_0, \ldots, x_n] \cos & [x_0, \ldots, x_n] \sin \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} [x_0, \ldots, x_{n-1}] f \\ [x_0, \ldots, x_n] f \end{pmatrix}.$$

Error formula

Assume that the Hermite interpolation problem at the extended sequence $x_0, \ldots, x_n$, $n \geq 1$, has a unique solution in C_n. Then

$$e(x) := f(x) - C(f; x_0, \ldots, x_n)(x) = e_2(x) - a_0 e_0(x) - a_1 e_1(x),$$

where a_0 and a_1 are the constants defined above,
$e_0(x) := \cos(x) - P(\cos; x_0, \ldots, x_n)(x)$,
$e_1(x) := \sin(x) - P(\sin; x_0, \ldots, x_n)(x)$ and
$e_2(x) := f(x) - P(f; x_0, \ldots, x_n)(x)$.

A bound for the interpolation error is

$$|e(x)| \leq \frac{K_{n+1} + |a_0| + |a_1|}{(n+1)!} \prod_{i=0}^{n} |x - x_i|,$$

with $K_{n+1} := \max_{x \in [x_0, x_n]} |f^{(n+1)}(x)|$.

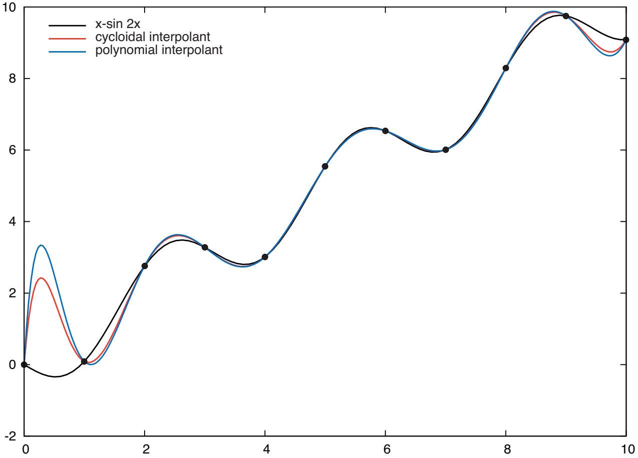
The constants a_0, a_1 can be bounded in terms of derivatives of f and the following error bound follows

$$|e(x)| \leq \frac{1}{(n+1)!}\left(K_{n+1} + 2\frac{K_{n-1}+K_n}{n!(n-1)!|D|}\right)\prod_{i=0}^{n}|x-x_i|,$$

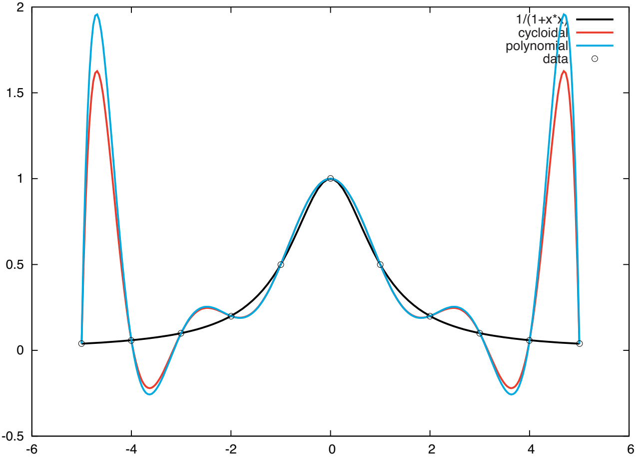
where $e(x) = f(x) - C(f; x_0, \ldots, x_n)(x)$, $K_j := \max_{x\in[x_0,x_n]} |f^{(j)}(x)|$, $j = n-1, n, n+1$, and

$$D = \left|\begin{array}{rr} [x_0, \ldots, x_{n-1}]\cos & [x_0, \ldots, x_{n-1}]\sin \\ [x_0, \ldots, x_n]\cos & [x_0, \ldots, x_n]\sin \end{array}\right|.$$

Example with a cycloidal interpolant in C_n

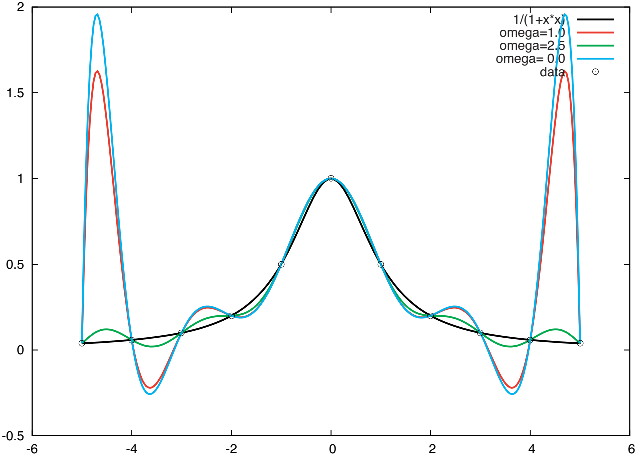


Runge example with a cycloidal interpolant in C_n



Runge example and choice of ω

Interpolate with spaces $\langle \cos(\omega x), \sin(\omega x), 1, x, \ldots, x^{n-2} \rangle$ and choose ω.



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The General Recurrence Relation for Divided Differences and the General Newton-Interpolation-Algorithm With Application to Trigonometric Interpolation
Numer. Math. **32** (1979), 393–408