A class of anisotropic multiple multiresolution analysis

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Jointly with:
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Description of expanding matrices and related objects
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Inside filterbanks and subdivisions
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Case study
Expanding matrices

Let \( M \in \mathbb{Z}^{s \times s} \) be an \textit{expanding matrix}, i.e.

- all its eigenvalues are larger than one in modulus
- \( \|M^{-n}\| \to 0 \)

\[
\downarrow
\]

as \( n \) increases, \( M^{-n}\mathbb{Z}^s \to \mathbb{R}^s \)
Expanding matrices

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- $\|M^{-n}\| \to 0$

as $n$ increases, $M^{-n}\mathbb{Z}^s \to \mathbb{R}^s$

- $M$ defines a sampling lattice
- $d = |\det(M)|$ is the number of cosets
The cosets have the form

\[ M\mathbb{Z}^s + \xi_j, \quad j = 0, \ldots, d - 1 \]

where

\[ \xi_j \in M[0, 1)^s \cap \mathbb{Z}^s \]

are the coset representatives.

It is well known that

\[ \mathbb{Z}^s = \bigcup_{j=0}^{d-1} (\xi_j + M\mathbb{Z}^s) \]
Separable/Nonseparable

\( \mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \)

\( \mathbf{M} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \)
Isotropy/Anisotropy

\[ M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \]

\[ M = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \]
Down/upsampling

Let $c \in \ell(\mathbb{Z}^s)$ be a given signal.
Down/upsampling

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- **Downsampling** operator $\downarrow_M$ associated to $M$:
  
  $$\downarrow_M c = c(M \cdot)$$
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- **Downsampling** operator $\downarrow_M$ associated to $M$:
  \[ \downarrow_M c = c(M \cdot) \]

- **Upsampling** operator $\uparrow_M$ associated to $M$:
  \[ \uparrow_M c(\alpha) = \begin{cases} 
  c(M^{-1}\alpha) & \text{if } \alpha \in M\mathbb{Z}^s \\
  0 & \text{otherwise}
\end{cases} \]
Filtering

Filter operator $F$:

$$Fc = f \ast c = \sum_{\alpha \in \mathbb{Z}^s} f(\cdot - \alpha)c(\alpha)$$

where $f = F\delta = (f(\alpha) : \alpha \in \mathbb{Z}^s)$ is the impulse response of $F$
\textit{d-channel filter bank}

Critically sampled: \( d = |\det M| \)
$d$-channel filter bank

Critically sampled: $d = |\det M|$

- **Analysis filter:**

$$F : \ell(\mathbb{Z}^s) \rightarrow \ell^d(\mathbb{Z}^s)$$

$$Fc = [\downarrow_M F_jc : j = 0, \ldots, d - 1]$$

- **Synthesis filter:**

$$G : \ell^d(\mathbb{Z}^s) \rightarrow \ell(\mathbb{Z}^s)$$

$$G [c_j : j = 0, \ldots, d - 1] = \sum_{j=0}^{d} G_j \uparrow_M c_j,$$
$d$-channel filter bank

Critically sampled: $d = |\det M|$

- **Analysis filter:**
  
  $$F : \ell(\mathbb{Z}^s) \to \ell^d(\mathbb{Z}^s)$$
  
  $$Fc = [\downarrow M F_j c : j = 0, \ldots, d - 1]$$

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  $$G : \ell^d(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$$
  
  $$G [c_j : j = 0, \ldots, d - 1] = \sum_{j=0}^{d} G_j \uparrow M c_j,$$

**Perfect reconstruction:**

$$GF = I$$
$d$-channel filter bank

By perfect reconstruction:

$$c \xrightarrow{F} \begin{bmatrix} c_0^1 \\ c_1^1 \\ \vdots \\ c_{d-1}^1 \end{bmatrix} = \begin{bmatrix} c_1^1 \\ \vdots \\ 0 \end{bmatrix} \xrightarrow{G} c$$

$F_0, G_0 \rightarrow$ low-pass

$F_j, G_j, \quad j > 0 \rightarrow$ high-pass

Multiresolution decomposition …
Iterated filter bank

MRA structure...

Multiple multiresolution analysis
Observe that

\[ G_j \uparrow c = g_j \ast \uparrow_M c = \sum_{\alpha \in \mathbb{Z}^s} g_j(\cdot - M\alpha) c(\alpha), \]

i.e. all reconstruction filters act as stationary subdivision operators with dilation matrix \( M \).
Stationary subdivision

Subdivision operator:

\[ S := S_{a, M} : \ell(\mathbb{Z}^s) \rightarrow \ell(\mathbb{Z}^s) \]

defined by

\[ c^{(n+1)} := Sc^{(n)} = \sum_{\alpha \in \mathbb{Z}^s} a(\cdot - M\alpha)c^{(n)}(\alpha) \]

where \( M \in \mathbb{Z}^{s \times s} \) is expanding
Multiple subdivision

Consider a set of a finite number of dilation matrices

\((M_j : j \in \mathbb{Z}_m)\)

where \(\mathbb{Z}_m = \{0, \ldots, m - 1\}\) for \(m \in \mathbb{N}\).
Multiple subdivision

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- Associate a mask to each \(M_j\):

\[a_j \in \ell (\mathbb{Z}^s), \quad j \in \mathbb{Z}_m\]
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- Associate a mask to each \( M_j \):
  \( a_j \in \ell(\mathbb{Z}^s), \quad j \in \mathbb{Z}_m \)

Together, \( a_j \) and \( M_j \) define \( m \) stationary subdivision operators

\( S_j := S_{a_j, M_j} \)
Multiple subdivision

Call

\[ \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}_m^n \]

a digit sequence of length \( n =: |\epsilon| \).
Multiple subdivision

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$$\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}_m^n$$

a digit sequence of length $n =: |\epsilon|$.

We collect all finite digit sequences in

$$\mathbb{Z}_m^* := \bigcup_{n \in \mathbb{N}} \mathbb{Z}_m^n$$

and extend $|\epsilon|$ canonically to $\epsilon \in \mathbb{Z}_m^*$.
Multiple subdivision

Consider the subdivision operator:

$$S_\epsilon = S_{\epsilon_n} \cdots S_{\epsilon_1}.$$
Multiple subdivision

Consider the subdivision operator:

\[ S_\epsilon = S_{\epsilon_n} \cdots S_{\epsilon_1}. \]

For any \( \epsilon \in \mathbb{Z}_m^* \) there exists a mask

\[ a_\epsilon = S_\epsilon \delta \]

such that

\[ S_\epsilon c = \sum_{\alpha \in \mathbb{Z}^s} a_\epsilon (\cdot - M_\epsilon \alpha) c(\alpha), \quad c \in \ell(\mathbb{Z}^s), \]

where

\[ M_\epsilon := M_{\epsilon_n} \cdots M_{\epsilon_1}, \quad n = |\epsilon|. \]
Multiple subdivision

Values of $S_\epsilon c = \text{approximations to a function on } M_\epsilon^{-1} \mathbb{Z}^s$. 
Multiple subdivision

Values of $S_\epsilon c$ = approximations to a function on $M_\epsilon^{-1} \mathbb{Z}^s$.

In order for $M_\epsilon^{-1} \mathbb{Z}^s$ to tend to $\mathbb{R}^s$:

Each matrix $M_j$ must be expanding,

All the matrices $M_\epsilon$ must be expanding

⇒ The matrices $M_\epsilon$ must all be jointly expanding i.e.

$$\lim_{|\epsilon| \to \infty} \|M_\epsilon^{-1}\| = 0,$$

or, equivalently,

$$\rho(M_\epsilon^{-1} : j \in \mathbb{Z}) < 1$$

(joint spectral radius condition)
Multiple subdivision

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Values of $S_\epsilon \subset \approx$ approximations to a function on $M_\epsilon^{-1} \mathbb{Z}^s$.

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- each matrix $M_j$ must be expanding,
- all the matrices $M_\epsilon$ must be expanding

\[\downarrow\]

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or, equivalently,

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\rho \left( M_j^{-1} : j \in \mathbb{Z}_m \right) < 1
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(joint spectral radius condition)
Multiple subdivision

Example: adaptive subdivision/discrete shearlets

Based on:
Multiple subdivision

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Based on:

- parabolic scaling \[
\begin{bmatrix}
2 \\
4
\end{bmatrix}
\]
Multiple subdivision

Example: adaptive subdivision/discrete shearlets

Based on:

- parabolic scaling $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- shear $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
Multiple subdivision

Example: adaptive subdivision/discrete shearlets

Based on:

- parabolic scaling \[
\begin{bmatrix}
2 \\
4
\end{bmatrix}
\]

- shear \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]

What about other choices? Case study ...
Multiple $d$-channel filter bank

For each $k \in \mathbb{Z}_m$

- **Analysis filters:** $F_k : \ell(\mathbb{Z}^s) \rightarrow \ell^d(\mathbb{Z}^s)$ acting as

  $$F_k c = \left[ \Downarrow M_k F_{k,j} c : j = 0, \ldots, d - 1 \right]$$
Multiple \(d\)-channel filter bank

For each \(k \in \mathbb{Z}_m\)

- **Analysis filters**: \(F_k : \ell(\mathbb{Z}^s) \to \ell^d(\mathbb{Z}^s)\) acting as

\[
F_k c = [\downarrow_{M_k} F_{k,j} c : j = 0, \ldots, d - 1]
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- **Synthesis filters**: \(G_k : \ell^d(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)\), acting as

\[
G_k [c_j : j = 0, \ldots, d - 1] = \sum_{j=0}^{d} G_{k,j} \uparrow_{M_k} c_j,
\]
Multiple $d$-channel filter bank

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  $$G_k [c_j : j = 0, \ldots, d-1] = \sum_{j=0}^{d} G_{k,j} \uparrow M_k c_j,$$

**Perfect reconstruction:**

$$G_k F_k = I, \quad k \in \mathbb{Z}_m$$
Multiple multiresolution analysis
Symbol notation

Given a finitely supported $a$

- **Symbol:**
  
  $$a^\#(z) := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) z^\alpha$$
Symbol notation

Given a finitely supported $a$

- **Symbol:**
  \[ a^\#(z) := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) z^\alpha \]

- **Subsymbols:**
  \[ a_{\xi_j}^\#(z) := \sum_{\alpha \in \mathbb{Z}^s} a(M\alpha + \xi_j) z^\alpha, \quad j = 0, \ldots, d - 1 \]
Filter bank construction

Start from the lowpass reconstruction filter $G_0$ associated to a mask $a$. 
Filter bank construction

Start from the **lowpass** reconstruction filter $G_0$ associated to a mask $a$.

$G_0$ can be completed to a perfect reconstruction filter bank if and only if $a$ is unimodular:

- algebraic property
- involved in general
- simple for **interpolatory** schemes
Filter bank construction

Start from the lowpass reconstruction filter $G_0$ associated to a mask $a$.

$G_0$ can be completed to a perfect reconstruction filter bank if and only if $a$ is unimodular:

- algebraic property
- involved in general
- simple for interpolatory schemes

In 1D $\rightarrow a^\#(z)$ and $a^\#(-z)$ have no common zeros.
Filter bank construction

Simplest filter bank $\longrightarrow$ lazy filters: translation operators

$$\tau_{\xi_i}, \quad i = 0, \ldots, d - 1$$

In fact

$$I = \sum_{i=0}^{d-1} \tau_{\xi_i} \uparrow \downarrow \tau_{-\xi_i},$$

Multiple multiresolution analysis
Filter bank construction

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In fact

\[ I = \sum_{i=0}^{d-1} \tau_{\xi_i} \uparrow \downarrow \tau_{-\xi_i}, \]

It:

- decomposes a signal modulo \(M\) in the analysis
- recombines the components in the synthesis
Filter bank construction

If $a$ defines an interpolatory subdivision scheme, then $G_0$ can be easily completed to a perfect reconstruction filter bank.
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A subdivision operator $S_a$ with dilation matrix $M$ is called **interpolatory** if

$$S_a c(M \cdot) = c, \quad \text{for any } c \in \ell(\mathbb{Z}^s)$$
The completion of an interpolatory \( a \) yields the prediction–correction scheme.
Prediction–correction scheme

The completion of an interpolatory analysis yields the prediction–correction scheme

- **Analysis part:**
  
  \[ F_0 = I, \quad F_j = \tau_{-\xi_j} (I - S_a \downarrow M), \quad j = 1, \ldots, d - 1, \]

- **Synthesis part:**
  
  \[ G_0 \quad \text{and} \quad G_j = \tau_{\xi_j}, \quad j = 1, \ldots, d - 1. \]
Prediction–correction scheme

In terms of symbols:

\[ F_0^\#(z) = 1, \quad F_j^\#(z) = z^{\xi_j} - a_{\xi_j}^\#(z^{-M}), \quad j = 1, \ldots, d - 1 \]

\[ G_0^\#(z) = a^\#(z), \quad F_j^\#(z) = z^{\xi_j}, \quad j = 1, \ldots, d - 1 \]
A special construction of $s$-variate interpolatory schemes

Let

$$M = \Theta \Sigma \Theta'$$

be a Smith factorization of the expanding matrix $M$, where

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_s \end{bmatrix}$$

and $\Theta, \Theta'$ unimodular.
A special construction of $s$-variate interpolatory schemes

1. Find $s$ univariate interpolatory subdivision schemes $b_j, \quad j = 1, \ldots, s$

with scaling factors or “arity” $\sigma_j$;
A special construction of $s$-variate interpolatory schemes

1. Find $s$ univariate interpolatory subdivision schemes

$$b_j, \quad j = 1, \ldots, s$$

with scaling factors or “arity” $\sigma_j$;

2. Consider the tensor product

$$b_\Sigma := \bigotimes_{j=1}^{s} b_j, \quad b_\Sigma(\alpha) = \prod_{j=1}^{s} b_j(\alpha_j), \quad \alpha \in \mathbb{Z}^s,$$

which is an interpolatory subdivision scheme for the diagonal scaling matrix $\Sigma$, i.e.

$$b_\Sigma(\Sigma \cdot) = \delta$$

Multiple multiresolution analysis
A special construction of $s$-variate interpolatory schemes

Set

$$b_M := b_\Sigma(\Theta^{-1}).$$
A special construction of $s$-variate interpolatory schemes

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Then:

$b_M$ defines an interpolatory scheme for the dilation matrix $M$.  

Multiple multiresolution analysis
A special construction of $s$-variate interpolatory schemes

Set

$$b_M := b_{\Sigma}(\Theta^{-1} \cdot)$$

Then:

$b_M$ defines an interpolatory scheme for the dilation matrix $M$.

In terms of symbols:

$$b^\#: M(z) = b^\#: (z^\Theta)$$
A special choice of scaling matrices

We are considering the matrices

\[ M_0 := \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \]

\[ M_1 := S_1 M_0 = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}, \]

where we make use of the shear matrices

\[ S_j := \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}, \quad j \in \mathbb{Z}. \]
A special choice of scaling matrices

It is easily verified that

\[ \det M_0 = \det M_1 = -3 \]
A special choice of scaling matrices

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- \( \det M_0 = \det M_1 = -3 \)
- \( M_0 \) is anisotropic (eigenvalues: \( \frac{1}{2} (1 \pm \sqrt{13}) \))

\( M_0 \) and \( M_1 \) are jointly expanding so they define a reasonable subdivision scheme.
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- $\det M_0 = \det M_1 = -3$
- $M_0$ is anisotropic (eigenvalues: $\frac{1}{2} (1 \pm \sqrt{13})$)
- $M_1$ is isotropic (eigenvalues: $\pm \sqrt{3}$)
A special choice of scaling matrices

It is easily verified that

- \( \det M_0 = \det M_1 = -3 \)
- \( M_0 \) is anisotropic (eigenvalues: \( \frac{1}{2} \left( 1 \pm \sqrt{13} \right) \))
- \( M_1 \) is isotropic (eigenvalues: \( \pm \sqrt{3} \))
- \( M_0 \) and \( M_1 \) are jointly expanding so they define a reasonable subdivision scheme.
Coset representation: $M_0$
Coset representation: $M_1$
The subdivision process

Sequence 0 0 0 0 0 0

Multiple multiresolution analysis
The subdivision process

Sequence 1 1 1 1 1 1

Initial data

Multiple multiresolution analysis
The subdivision process

Sequence 0 1 0 1 0 1

Initial data

Multiple multiresolution analysis
The subdivision process

Sequence 1 0 1 0 1 0

Initial data

M0

M0 M1 M0

M0 M1 M0 M1 M0

M1 M0 M1 M0 M1 M0

Multiple multiresolution analysis
In "Multiple MRA" one considers functions of the form

\[ \phi_\eta(M_\epsilon \cdot -\alpha), \quad \alpha \in \mathbb{Z}^s. \]
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- Role of $M_\epsilon$: scale & rotate
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Can we get "all rotations" by appropriate \( \epsilon \)?
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Can we get "all rotations" by appropriate $\epsilon$?

→ Slope resolution
Slope resolution

Action of:

\[ M_1 M_1 \text{ (blue)}, \ M_0 M_1 \text{ (red)}, \ M_1 M_0 \text{ (green)}, \ M_0 M_0 \text{ (cyan)} \]
on the unit vectors
Slope resolution

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\]

on the unit vectors
Slope resolution

Can all directions, i.e., all lines through the origin, be generated by applying an appropriate $M_\epsilon$ to a given reference line?
Slope resolution

Given the reference line

\[ L_x := \mathbb{R} x, \quad x \in \mathbb{R}^2 \]

and a target line

\[ L_y := \mathbb{R} y, \quad y \in \mathbb{R}^2 \]

we ask whether there exists \( \epsilon \in \mathbb{Z}^*_m \) such that

\[ L_y \sim M_\epsilon L_x. \]
Slope resolution

We represent lines by means of slopes, setting

\[ L(s) := \mathbb{R} \begin{bmatrix} 1 \\ s \end{bmatrix}, \quad s \in \mathbb{R} \cup \{\pm \infty\}, \]

where \( s = \pm \infty \) corresponds to (the same) vertical line.
Slope resolution

We represent lines by means of slopes, setting

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where \( s = \pm \infty \) corresponds to (the same) vertical line.

**Theorem**

*For each \( s \in (0, \frac{1}{2}) \), any \( s' \in \mathbb{R} \) and any \( \delta > 0 \) there exists \( \epsilon \in \mathbb{Z}_m^* \) such that*

\[ |s' - s_{\epsilon}| < \delta, \quad L(s_{\epsilon}) = M_{\epsilon} L_s. \]
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**Theorem**

*For each \( s \in (0, \frac{1}{2}) \), any \( s' \in \mathbb{R} \) and any \( \delta > 0 \) there exists \( \epsilon \in \mathbb{Z}_m^* \) such that

\[ |s' - s_\epsilon| < \delta, \quad L(s_\epsilon) = M_\epsilon L_s. \]

Indeed even combinations of \( M_{01} = M_0 M_1 \) and \( M_{01} = M_1 M_0 \) are sufficient to satisfy the claim of the theorem.*
Bivariate interpolatory schemes associated to $M_0$ and $M_1$

Smith factorizations of $M_0$, $M_1$:

$$
M_0 = \begin{bmatrix}
4 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
-1 & 3
\end{bmatrix}
\begin{bmatrix}
1 & -2 \\
-1 & 3
\end{bmatrix},
$$

$$
M_1 = \begin{bmatrix}
5 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
-1 & 3
\end{bmatrix}
\begin{bmatrix}
1 & -2 \\
-1 & 3
\end{bmatrix}.
$$
Bivariate interpolatory schemes associated to $M_0$ and $M_1$

Possible choices for the ternary interpolatory schemes

- piecewise linear interpolant:

$$b_2 = \frac{1}{3} (\ldots, 0, 1, 2, 3, 2, 1, 0, \ldots)$$
Bivariate interpolatory schemes associated to \( M_0 \) and \( M_1 \)

Possible choices for the ternary interpolatory schemes

- piecewise linear interpolant:
  \[
  b_2 = \frac{1}{3} (\ldots, 0, 1, 2, 3, 2, 1, 0, \ldots)
  \]

- four point scheme based on local cubic interpolation
  \[
  b_2 = \frac{1}{81} (\ldots, 0, -4, -5, 0, 30, 60, 81, 60, 30, 0, -5, -4, 0, \ldots)
  \]
Bivariate interpolatory schemes associated to $M_0$ and $M_1$

The schemes are obtained from

$$b^\#_M(z) = b^\#_\Sigma (z^\Theta)$$

which result in the following two symbols

$$A^\#_1(z_1, z_2) = \frac{z_1^{-2}}{3} \left(1 + z_1 + z_1^2\right)^2,$$

$$A^\#_2(z_1, z_2) = -\frac{z_1^{-5}}{81} \left(1 + z_1 + z_1^2\right)^4 \left(4z_1^2 - 11z_1 + 4\right),$$
Theorem
Suppose:

- $b_j, j = 1, \ldots, s$ define univariate subdivision schemes with scaling factors $\sigma_j \geq 1$
- $S_{b_j}1 = 1$. 

Then $M$ is a convergent subdivision scheme with dilation matrix $M$ iff the vector scheme $S_B \Sigma$ defined by

$$\nabla D (\Theta' \Theta) - 1 S_B \Sigma = S_B \Sigma \nabla$$

satisfies

$$1 > \rho_\infty (S_B \Sigma | \nabla) := \lim_{n \to \infty} \sup \| \nabla c \| \leq 1 \frac{\| S_n B \Sigma \nabla c \|}{n}.$$
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Suppose:

- $b_j$, $j = 1, \ldots, s$ define univariate subdivision schemes with scaling factors $\sigma_j \geq 1$
- $S_{b_j}1 = 1$.

Then $b_M$ is a convergent subdivision scheme with dilation matrix $M$ iff the vector scheme $S_{B_S}$ defined by
\[ \nabla D_{(\Theta', \Theta)^{-1}} S_{b_S} = S_{B_S} \nabla \]
satisfies
\[ 1 > \rho_{\infty} (S_{B_S} | \nabla) := \lim_{n \to \infty} \sup_{\|\nabla c\| \leq 1} \left\| S_{B_S}^n \nabla c \right\|^{1/n}. \]

where

- $D_{\Lambda}$ is the dilation operator $D_{\Lambda}c = c(\Lambda \cdot)$
- $\nabla$ is the forward difference operator
\[ \nabla c = [c(\cdot + \epsilon_j) - c : j = 1, \ldots, s] \]
$A^\#_1(z_1, z_2) = \frac{z_1^{-2}}{3} (1 + z_1 + z_1^2)^2$, $M_0 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$
\[ A^\#_1(z_1, z_2) = \frac{z_1^{-2}}{3} (1 + z_1 + z_1^2)^2, \quad M_1 = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \]
\[ A_2^\#(z_1, z_2) = -\frac{z_1^{-5}}{81} (1 + z_1 + z_1^2)^4 (4z_1^2 - 11z_1 + 4), \]
\[ M_0 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \]
\[ A_2^\#(z_1, z_2) = -\frac{z_1^{-5}}{81} (1 + z_1 + z_1^2)^4 (4z_1^2 - 11z_1 + 4), \]
\[ M_1 = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \]
Filter bank associated to $M_0$

$$A^h_1(z_1, z_2) = \frac{z_1^{-2}}{3} (1 + z_1 + z_1^2)^2$$ and $M_0$

Analysis

$$F_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad F_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad F_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{2}{3} & 0
\end{bmatrix}$$

Synthesis

$$G_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad G_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad G_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}$$
Filter bank associated to $M_1$

\[ A_1^\#(z_1, z_2) = \frac{z_1^{-2}}{3} (1 + z_1 + z_1^2)^2 \text{ and } M_1 \]

Analysis

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & 1 & -\frac{2}{3} & 0 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Synthesis

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Grazie!