A class of anisotropic multiple multiresolution analysis

Mariantonia Cotronei

University of Reggio Calabria, Italy

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Jointly with:

Mira Bozzini, Milvia Rossini, Tomas Sauer

- Description of expanding matrices and related objects

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Expanding matrices

Let $M \in \mathbb{Z}^{s \times s}$ be an expanding matrix, i.e.

- all its its eigenvalues are larger than one in modulus
- $\|M^{-n}\| \to 0$

$$\Downarrow$$

as n increases, $M^{-n}\mathbb{Z}^s \to \mathbb{R}^s$

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- M defines a sampling lattice
- $d = |\det(M)|$ is the number of cosets

The cosets have the form

$$M\mathbb{Z}^s + \xi_j, \quad j = 0, \ldots, d-1$$

where

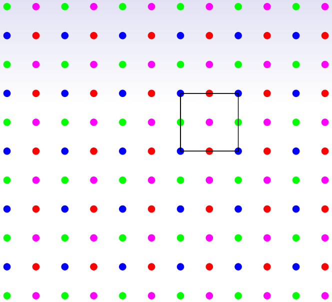
$$\xi_j \in M[0,1)^s \bigcap \mathbb{Z}^s$$

are the coset representatives.

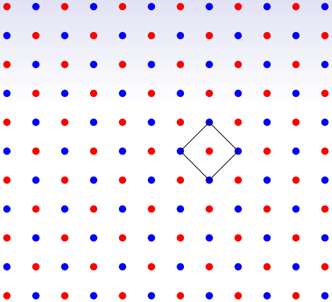
It is well known that

$$\mathbb{Z}^s = \bigcup_{j=0}^{d-1} (\xi_j + M\mathbb{Z}^s)$$

Separable/Nonseparable

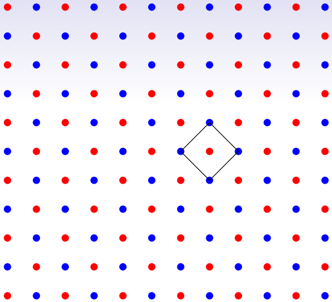


$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Isotropy/Anisotropy



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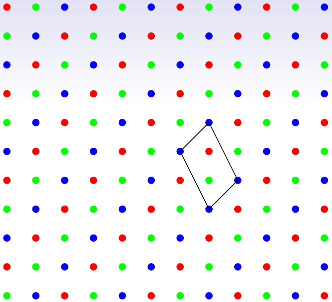

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Down/upsampling

Let $c \in \ell(\mathbb{Z}^s)$ be a given signal.

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- Upsampling operator $\uparrow_M$ associated to M:

$$\uparrow_M c(\alpha) = \begin{cases} c(M^{-1}\alpha) & \text{if } \alpha \in M\mathbb{Z}^s \\ 0 & \text{otherwise} \end{cases}$$

Filtering

- Filter operator F:

$$Fc = f * c = \sum_{\alpha \in \mathbb{Z}^s} f(\cdot - \alpha) c(\alpha)$$

where $f = F\delta = (f(\alpha) : \alpha \in \mathbb{Z}^s)$ is the impulse response of F

d-channel filter bank

Critically sampled: $d = |\det M|$

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- Analysis filter:

$$F : \ell(\mathbb{Z}^s) \to \ell^d(\mathbb{Z}^s)$$

$$Fc = [\downarrow_M F_j c \;:\; j = 0, \ldots, d-1]$$

- Synthesis filter:

$$G : \ell^d(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$$

$$G\,[c_j \;:\; j = 0, \ldots, d-1] = \sum_{j=0}^{d} G_j \uparrow_M c_j,$$

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Perfect reconstruction:

$$GF = I$$

d-channel filter bank

By perfect reconstruction:

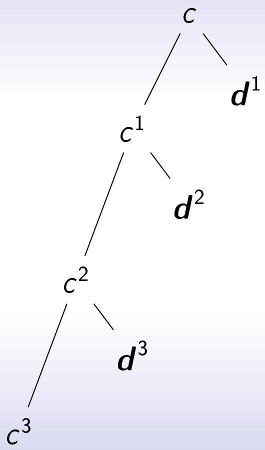
$$c \xrightarrow{F} \left[\begin{array}{c} c_0^1 \\ \hline c_1^1 \\ \vdots \\ c_{d-1}^1 \end{array}\right] = \left[\begin{array}{c} c^1 \\ \hline \boldsymbol{d}^1 \end{array}\right] \xrightarrow{G} c$$

$F_0, G_0 \longrightarrow$ low-pass
$F_j, G_j, \quad j > 0 \longrightarrow$ high-pass

Multiresolution decomposition ...

Iterated filter bank

MRA structure...



Observe that

$$G_j \uparrow c = g_j * \uparrow_M c = \sum_{\alpha \in \mathbb{Z}^s} g_j(\cdot - M\alpha)\, c(\alpha),$$

i.e. all reconstruction filters act as stationary subdivision operators with dilation matrix M.

Stationary subdivision

Subdivision operator:

$$S := S_{a,M} \colon \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$$

defined by

$$c^{(n+1)} := Sc^{(n)} = \sum_{\alpha \in \mathbb{Z}^s} a(\cdot - M\alpha) c^{(n)}(\alpha)$$

where $M \in \mathbb{Z}^{s \times s}$ is expanding

Multiple subdivision

- Consider a set of a finite number of dilation matrices

$$(M_j \ : \ j \in \mathbb{Z}_m)$$

where $\mathbb{Z}_m = \{0, \ldots, m-1\}$ for $m \in \mathbb{N}$.

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$$a_j \in \ell(\mathbb{Z}^s), \quad j \in \mathbb{Z}_m$$

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$$a_j \in \ell\left(\mathbb{Z}^s\right), \quad j \in \mathbb{Z}_m$$

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Together, a_j and M_j define m stationary subdivision operators

$$S_j := S_{a_j, M_j}$$

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We collect all finite digit sequences in

$$\mathbb{Z}_m^* := \bigcup_{n \in \mathbb{N}} \mathbb{Z}_m^n$$

and extend $|\epsilon|$ canonically to $\epsilon \in \mathbb{Z}_m^*$.

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$$S_\epsilon = S_{\epsilon_n} \cdots S_{\epsilon_1}.$$

For any $\epsilon \in \mathbb{Z}_m^*$ there exists a mask

$$a_\epsilon = S_\epsilon \delta$$

such that

$$S_\epsilon c = \sum_{\alpha \in \mathbb{Z}^s} a_\epsilon (\cdot - M_\epsilon \alpha)\, c(\alpha), \qquad c \in \ell(\mathbb{Z}^s),$$

where

$$M_\epsilon := M_{\epsilon_n} \cdots M_{\epsilon_1}, \qquad n = |\epsilon|.$$

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The matrices M_ϵ must all be jointly expanding i.e.

$$\lim_{|\epsilon|\to\infty} \left\| M_\epsilon^{-1} \right\| = 0, \tag{1}$$

or, equivalently,

$$\rho\left(M_j^{-1}\ :\ j \in \mathbb{Z}_m\right) < 1$$

(joint spectral radius condition)

Multiple subdivision

Example: adaptive subdivision/discrete shearlets

Based on:

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What about other choices?
Case study ...

Multiple d-channel filter bank

For each $k \in \mathbb{Z}_m$

- Analysis filters: $F_k : \ell(\mathbb{Z}^s) \to \ell^d(\mathbb{Z}^s)$ acting as

$$F_k c = [\downarrow_{M_k} F_{k,j} c \ : \ j = 0, \dots, d-1]$$

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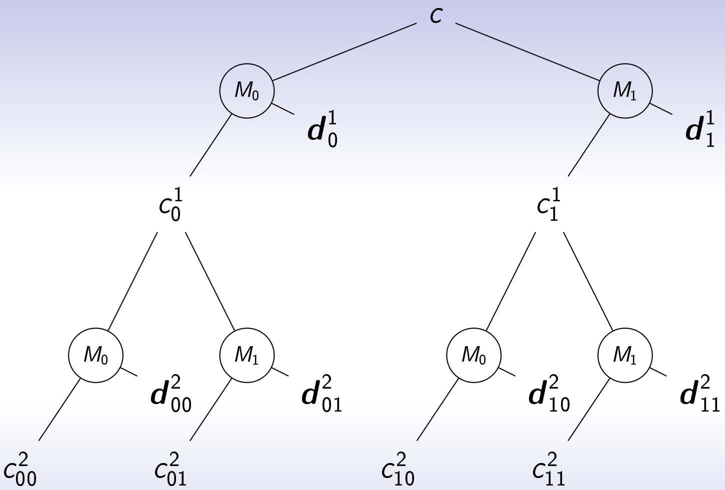
Perfect reconstruction:

$$G_k F_k = I, \quad k \in \mathbb{Z}_m$$


c
M_0
M_1
$\boldsymbol{d}_0^1$
$\boldsymbol{d}_1^1$
c_0^1
c_1^1
M_0
M_1
M_0
M_1
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$\boldsymbol{d}_{01}^2$
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c_{00}^2
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Symbol notation

Given a finitely supported a

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- Subsymbols:

$$a^\sharp_{\xi_j}(z) := \sum_{\alpha\in\mathbb{Z}^s} a(M\alpha+\xi_j) z^\alpha, \quad j = 0,\dots,d-1$$

Filter bank construction

Start from the **lowpass** reconstruction filter G_0 associated to a mask a.

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G_0 can be completed to a perfect reconstruction filter bank if and only if a is unimodular:

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In 1D $\longrightarrow$ $a^\sharp(z)$ and $a^\sharp(-z)$ have no common zeros.

Filter bank construction

Simplest filter bank $\longrightarrow$ lazy filters: translation operators

$$\tau_{\xi_i}, \quad i = 0, \ldots, d-1$$

In fact

$$I = \sum_{i=0}^{d-1} \tau_{\xi_i} \uparrow\downarrow \tau_{-\xi_i},$$

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In fact

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It:

- decomposes a signal modulo M in the analysis
- recombines the components in the synthesis

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If a defines an interpolatory subdivision scheme, then G_0 can be easily completed to a perfect reconstruction filter bank.

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A subdivision operator S_a with dilation matrix M is called **interpolatory** if

$$S_a c(M\cdot) = c, \qquad \text{for any } c \in \ell(\mathbb{Z}^s)$$

Prediction–correction scheme

The completion of an interpolatory a yields the
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- Analysis part:

$$F_0 = I, \qquad F_j = \tau_{-\xi_j}\left(I - S_a \downarrow_M\right), \quad j = 1, \ldots, d-1,$$

- Synthesis part:

$$G_0 \qquad \text{and} \qquad G_j = \tau_{\xi_j}, \quad j = 1, \ldots, d-1.$$

Prediction–correction scheme

In terms of symbols:

$$F_0^\sharp(z) = 1, \qquad F_j^\sharp(z) = z^{\xi_j} - a_{\xi_j}^\sharp(z^{-M}), \quad j = 1, \dots, d-1$$

$$G_0^\sharp(z) = a^\sharp(z), \qquad F_j^\sharp(z) = z^{\xi_j}, \quad j = 1, \dots, d-1$$

A special construction of s-variate interpolatory schemes

Let

$$M = \Theta \Sigma \Theta'$$

be a Smith factorization of the expanding matrix M, where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_s \end{bmatrix}$$

and Θ, Θ' unimodular

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1. Find s univariate interpolatory subdivision schemes

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with scaling factors or "arity" σ_j;

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with scaling factors or "arity" σ_j;

2. Consider the tensor product

$$b_\Sigma := \bigotimes_{j=1}^{s} b_j, \qquad b_\Sigma(\alpha) = \prod_{j=1}^{s} b_j(\alpha_j), \quad \alpha \in \mathbb{Z}^s,$$

which is an interpolatory subdivision scheme for the diagonal scaling matrix Σ, i.e.

$$b_\Sigma(\Sigma \cdot) = \delta$$

A special construction of s-variate interpolatory schemes

3. Set

$$b_M := b_\Sigma(\Theta^{-1}\cdot)$$

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In terms of symbols:

$$b_M^\sharp(z) = b_\Sigma^\sharp\left(z^\Theta\right)$$

A special choice of scaling matrices

We are considering the matrices

$$M_0 := \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$M_1 := S_1 M_0 = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix},$$

where we make use of the **shear matrices**

$$S_j := \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}, \qquad j \in \mathbb{Z}.$$

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It is easily verified that

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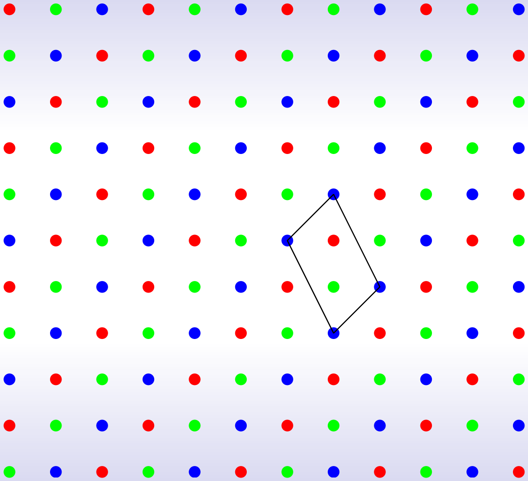
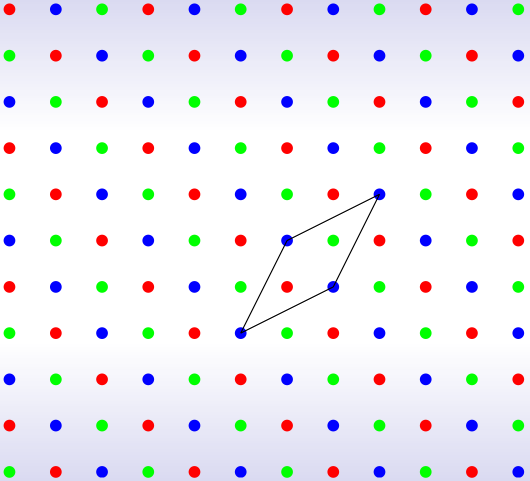
- $\det M_0 = \det M_1 = -3$
- M_0 is anisotropic (eigenvalues: $\frac{1}{2}\left(1 \pm \sqrt{13}\right)$)
- M_1 is isotropic (eigenvalues: $\pm\sqrt{3}$)
- M_0 and M_1 are jointly expanding so they define a reasonable subdivision scheme.

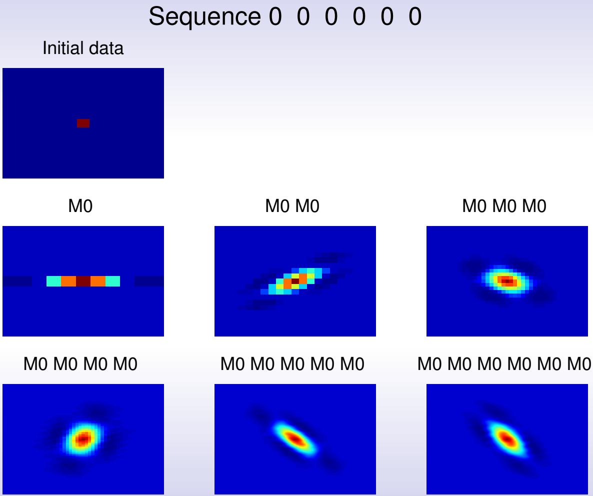
Coset representation: M_0



Coset representation: M_1



The subdivision process



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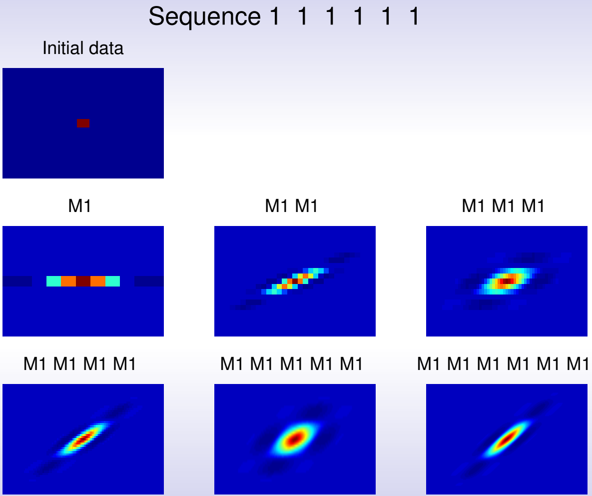
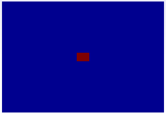


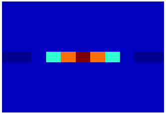
The subdivision process

Sequence 0 1 0 1 0 1

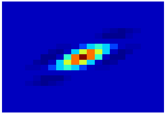
Initial data



M1



M0 M1



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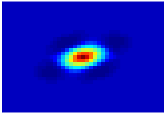

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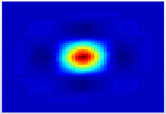

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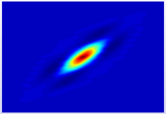
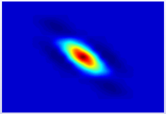


M0 M1 M0 M1 M0 M1



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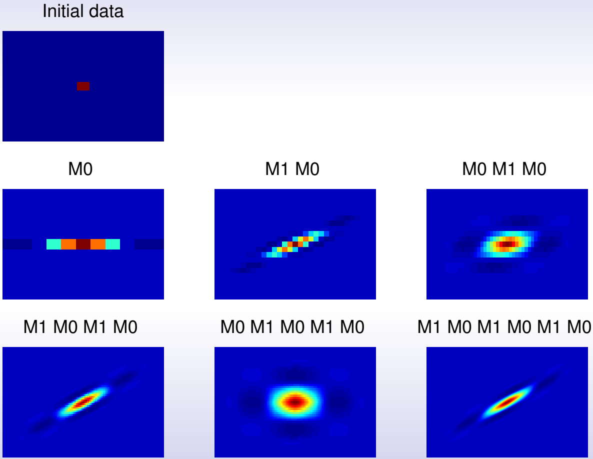

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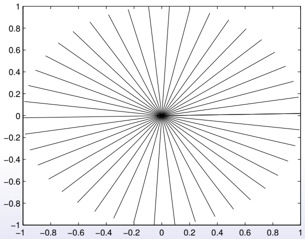
$\rightarrow$ Slope resolution

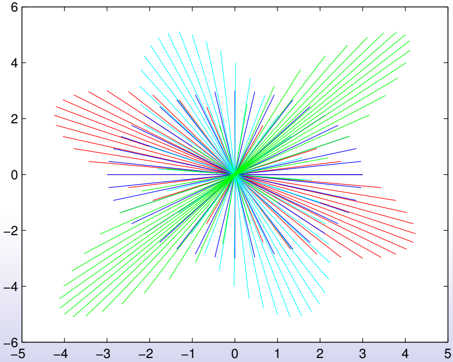
Slope resolution

Action of:

$M_1 M_1$ (blue), $M_0 M_1$ (red), $M_1 M_0$ (green), $M_0 M_0$ (cyan)

on the unit vectors





Slope resolution

Action of:

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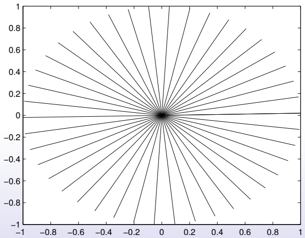
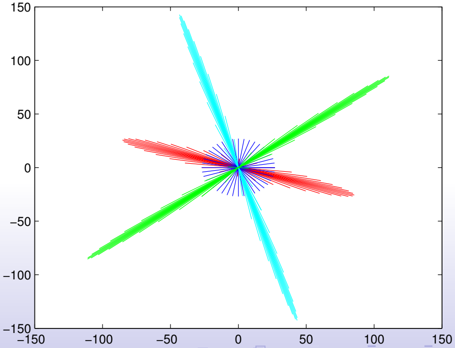
$$M_1\,M_0\,M_1\,M_0\,M_1\,M_0 \;\; (\text{green}),\;\; M_0\,M_0\,M_0\,M_0\,M_0\,M_0 \;\; (\text{cyan})$$

on the unit vectors





Slope resolution

Can all directions, i.e., all lines through the origin, be generated by applying an appropriate M_ϵ to a given reference line?

Slope resolution

Given the reference line

$$L_x := \mathbb{R}\, x, \quad x \in \mathbb{R}^2$$

and a target line

$$L_y := \mathbb{R}\, y, \quad y \in \mathbb{R}^2$$

we ask whether there exists $\epsilon \in \mathbb{Z}_m^*$ such that

$$L_y \sim M_\epsilon L_x.$$

Slope resolution

We represent lines by means of slopes, setting

$$L(s) := \mathbb{R} \begin{bmatrix} 1 \\ s \end{bmatrix}, \qquad s \in \mathbb{R} \cup \{\pm\infty\},$$

where $s = \pm\infty$ corresponds to (the same) vertical line.

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Theorem

For each $s \in (0, \frac{1}{2})$, any $s' \in \mathbb{R}$ and any $\delta > 0$ there exists $\epsilon \in \mathbb{Z}_m^*$ such that

$$|s' - s_\epsilon| < \delta, \qquad L(s_\epsilon) = M_\epsilon L_s.$$

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Indeed even combinations of $M_{01} = M_0 M_1$ and $M_{01} = M_1 M_0$ are sufficient to satisfy the claim of the theorem.

Bivariate interpolatory schemes associated to M_0 and M_1

Smith factorizations of M_0, M_1:

$$
\begin{aligned}
M_0 &= \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \\ & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, \\
M_1 &= \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \\ & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}.
\end{aligned}
$$

Bivariate interpolatory schemes associated to M_0 and M_1

Possible choices for the ternary interpolatory schemes

- piecewise linear interpolant:

$$b_2 = \frac{1}{3}(\dots, 0, 1, 2, 3, 2, 1, 0, \dots)$$

Bivariate interpolatory schemes associated to M_0 and M_1

Possible choices for the ternary interpolatory schemes

- piecewise linear interpolant:

$$b_2 = \frac{1}{3}(\ldots, 0, 1, 2, 3, 2, 1, 0, \ldots)$$

- four point scheme based on local cubic interpolation

$$b_2 = \frac{1}{81}(\ldots, 0, -4, -5, 0, 30, 60, 81, 60, 30, 0, -5, -4, 0, \ldots)$$

Bivariate interpolatory schemes associated to M_0 and M_1

The schemes are obtained from

$$b^{\sharp}_{M}(z) = b^{\sharp}_{\Sigma}\left(z^{\Theta}\right)$$

which result in the following two symbols

$$A^{\sharp}_{1}(z_1, z_2) = \frac{z_1^{-2}}{3}\left(1 + z_1 + z_1^2\right)^2 ,$$

$$A^{\sharp}_{2}(z_1, z_2) = -\frac{z_1^{-5}}{81}\left(1 + z_1 + z_1^2\right)^4 \left(4z_1^2 - 11z_1 + 4\right) ,$$

Theorem

Suppose:

- b_j, $j = 1, \ldots, s$ *define univariate subdivision schemes with scaling factors* $\sigma_j \geq 1$
- $S_{b_j} 1 = 1$.

Theorem

Suppose:

- b_j, $j = 1, \ldots, s$ *define univariate subdivision schemes with scaling factors* $\sigma_j \geq 1$
- $S_{b_j} 1 = 1$.

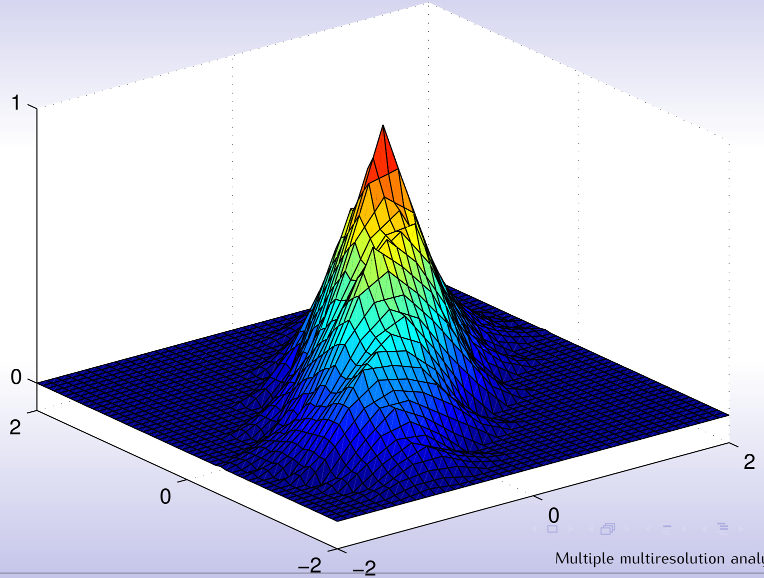
Then b_M *is a convergent subdivision scheme with dilation matrix* M *iff the vector scheme* S_{B_Σ} *defined by* $\nabla D_{(\Theta'\Theta)^{-1}} S_{b_\Sigma} = S_{B_\Sigma} \nabla$ *satisfies*

$$1 > \rho_\infty \left(S_{B_\Sigma} \mid \nabla \right) := \lim_{n \to \infty} \sup_{\|\nabla c\| \leq 1} \left\| S_{B_\Sigma}^n \nabla c \right\|^{1/n}.$$

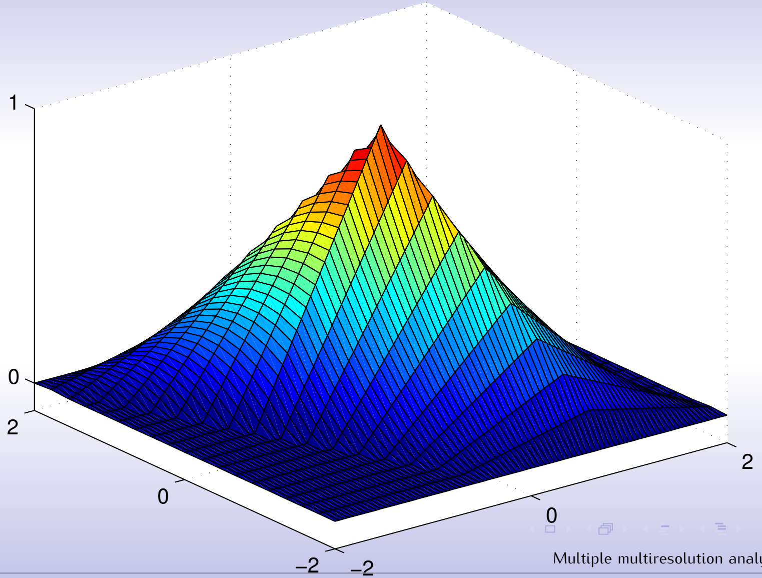
where

- D_Λ *is the* ***dilation operator*** $D_\Lambda c = c(\Lambda \cdot)$
- ∇ *is the* ***forward difference operator*** $\nabla c = [c(\cdot + \epsilon_j) - c \;:\; j = 1, \ldots, s]$

$$A_1^\sharp(z_1, z_2) = \frac{z_1^{-2}}{3}\left(1 + z_1 + z_1^2\right)^2, \; M_0 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

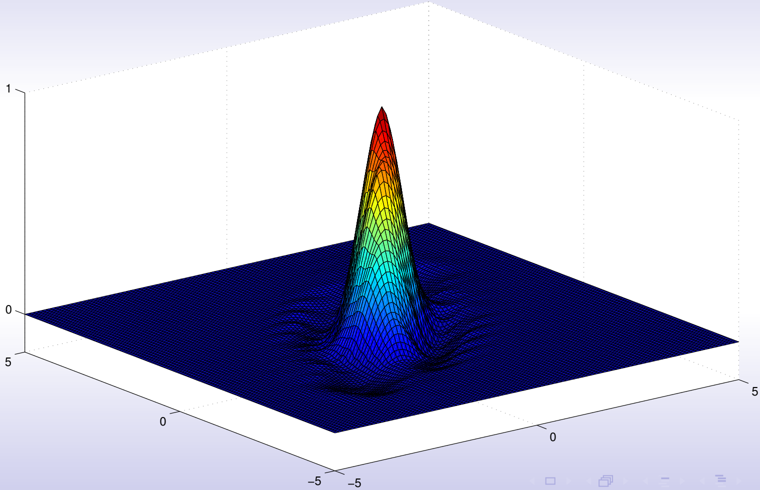


$$A_1^\sharp(z_1, z_2) = \frac{z_1^{-2}}{3}\left(1 + z_1 + z_1^2\right)^2, \; M_1 = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$$



$$A_2^\sharp(z_1, z_2) = -\frac{z_1^{-5}}{81}(1 + z_1 + z_1^2)^4\ (4z_1^2 - 11z_1 + 4),$$

$$M_0 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$



$$A_2^\sharp(z_1, z_2) = -\frac{z_1^{-5}}{81}\,(1 + z_1 + z_1^2)^4\;(4z_1^2 - 11z_1 + 4),$$

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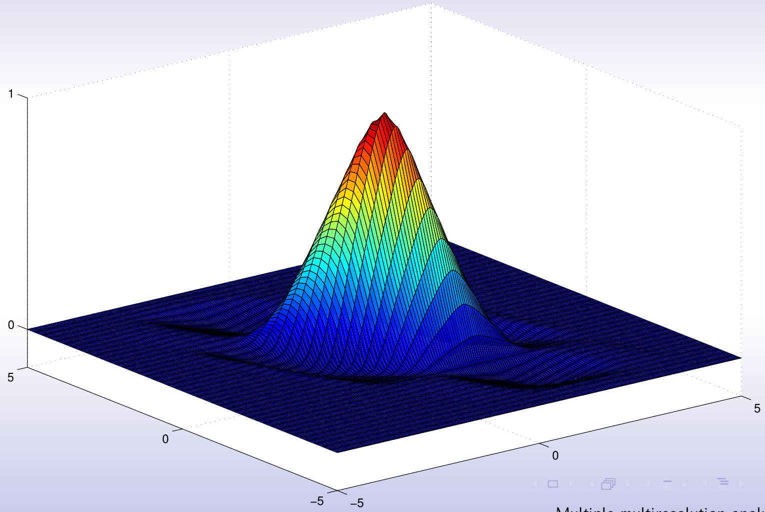


Filter bank associated to M_0

$A_1^\sharp(z_1, z_2) = \frac{z_1^{-2}}{3}(1 + z_1 + z_1^2)^2$ and M_0

Analysis

$$F_0 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad F_1 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad F_2 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{2}{3} & 0 \end{bmatrix}$$

Synthesis

$$G_0 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad G_1 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad G_2 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Filter bank associated to M_1

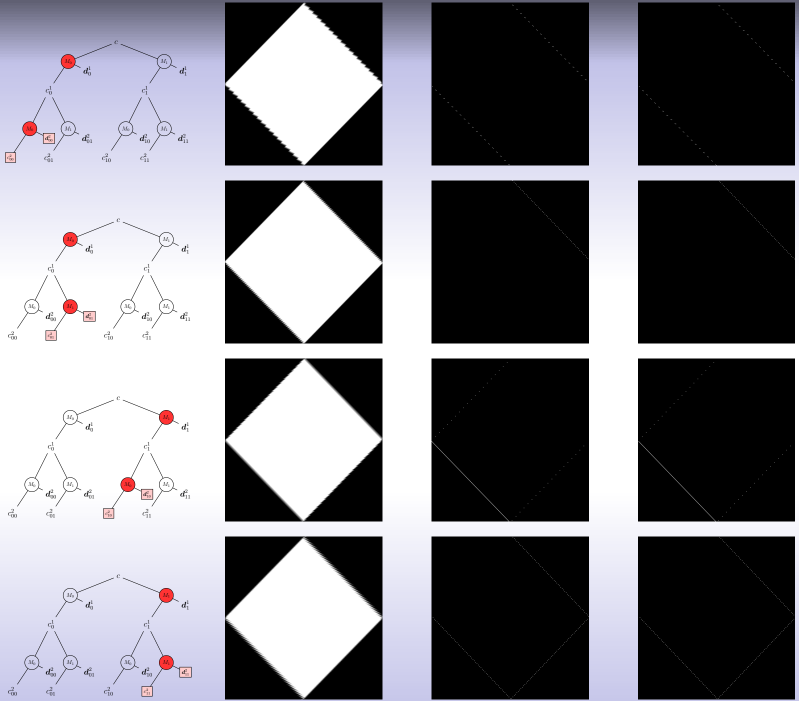
$$A_1^\sharp(z_1, z_2) = \frac{z_1^{-2}}{3}(1 + z_1 + z_1^2)^2 \text{ and } M_1$$

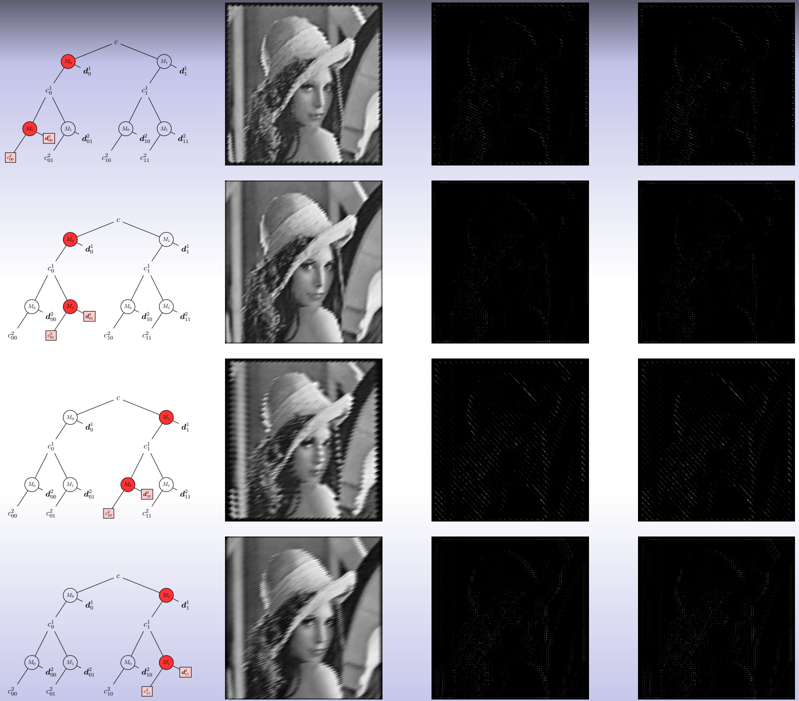
Analysis

$$F_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{2}{3} & 0 \end{bmatrix} \quad F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Synthesis

$$G_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$





c
M0
M1
d_0^1
d_1^1
c_0^1
c_1^1
M0
M1
M0
M1
d_00^2
d_01^2
d_10^2
d_11^2
c_00^2
c_01^2
c_10^2
c_11^2

Grazie!