Bijective Composite Mean Value Mappings

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joint work with

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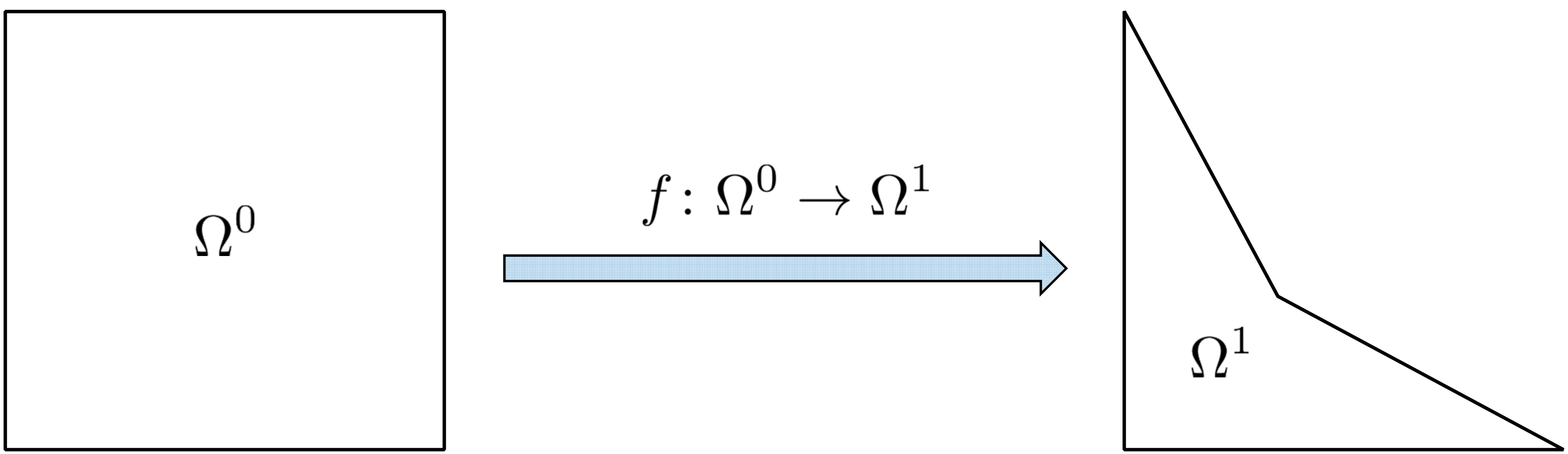
Introduction

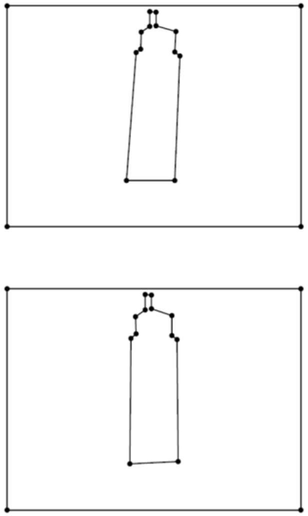
- special bivariate interpolation problem
- find mapping f between two simple polygons
  - bijective
  - linear along edges



Motivation

- image warping

original image



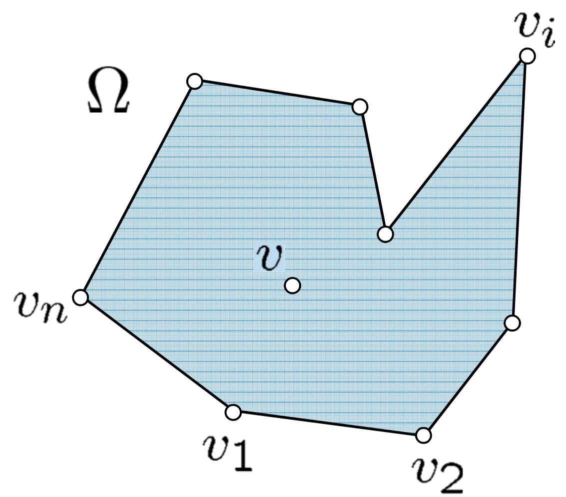
mask

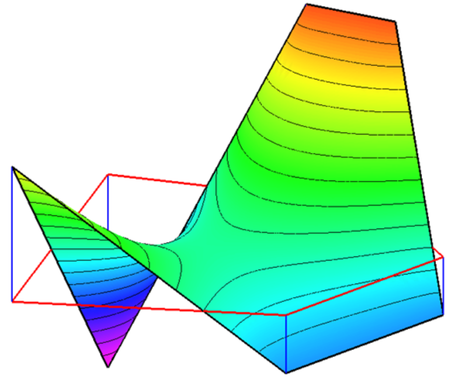
warped image

Barycentric coordinates

- functions $b_i : \Omega \to \mathbb{R}$ with
  - partition of unity $$\sum_{i=1}^{n} b_i(v) = 1$$
  - linear reproduction $$\sum_{i=1}^{n} b_i(v) v_i = v$$
  - Lagrange property $$b_i(v_j) = \delta_{ij}$$
- interpolation of data f_i given at v_i

$$f(v) = \sum_{i=1}^{n} b_i(v) f_i$$

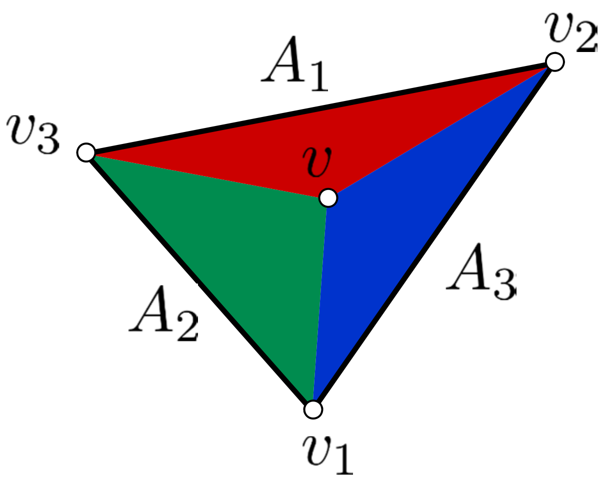




Barycentric coordinates

- special case $n = 3$

$$b_i(v) = \frac{A_i(v)}{A_1(v) + A_2(v) + A_3(v)}$$



- general case
  - homogeneous weight functions $w_i : \Omega \to \mathbb{R}$ with

$$\sum_{i=1}^{n} w_i(v)(v_i - v) = 0$$

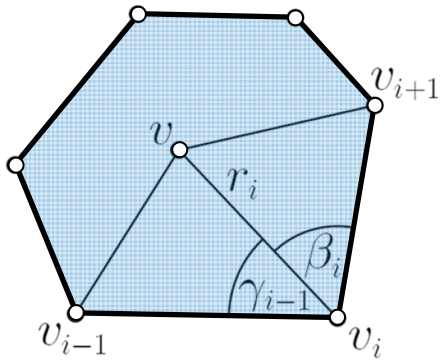
  - barycentric coordinates

$$b_i(v) = \frac{w_i(v)}{w_1(v) + \cdots + w_n(v)}$$

Examples

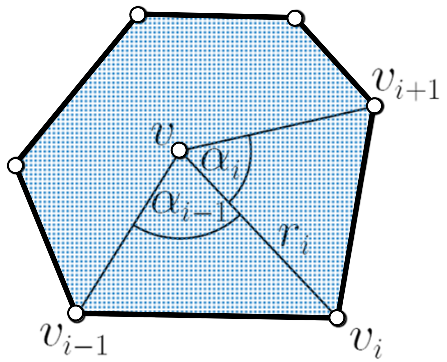
- Wachspress (WP) coordinates

$$w_i = \frac{\cot \gamma_{i-1} + \cot \beta_i}{{r_i}^2}$$



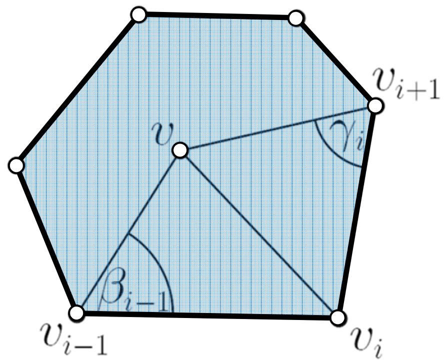
- mean value (MV) coordinates

$$w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{r_i}$$



- discrete harmonic (DH) coordinates

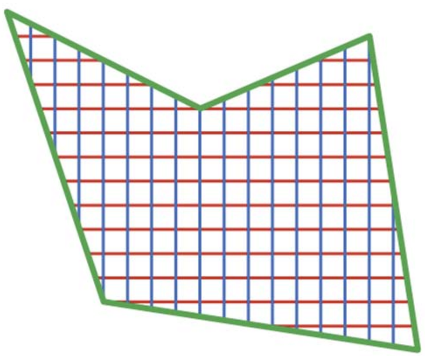
$$w_i = \cot \beta_{i-1} + \cot \gamma_i$$



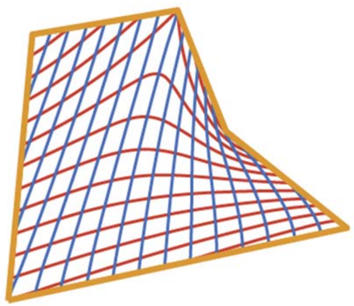
Barycentric mappings

source polygon

target polygon



$$f(v) = \sum_{j=1}^{n} b_j^0(v) v_j^1$$

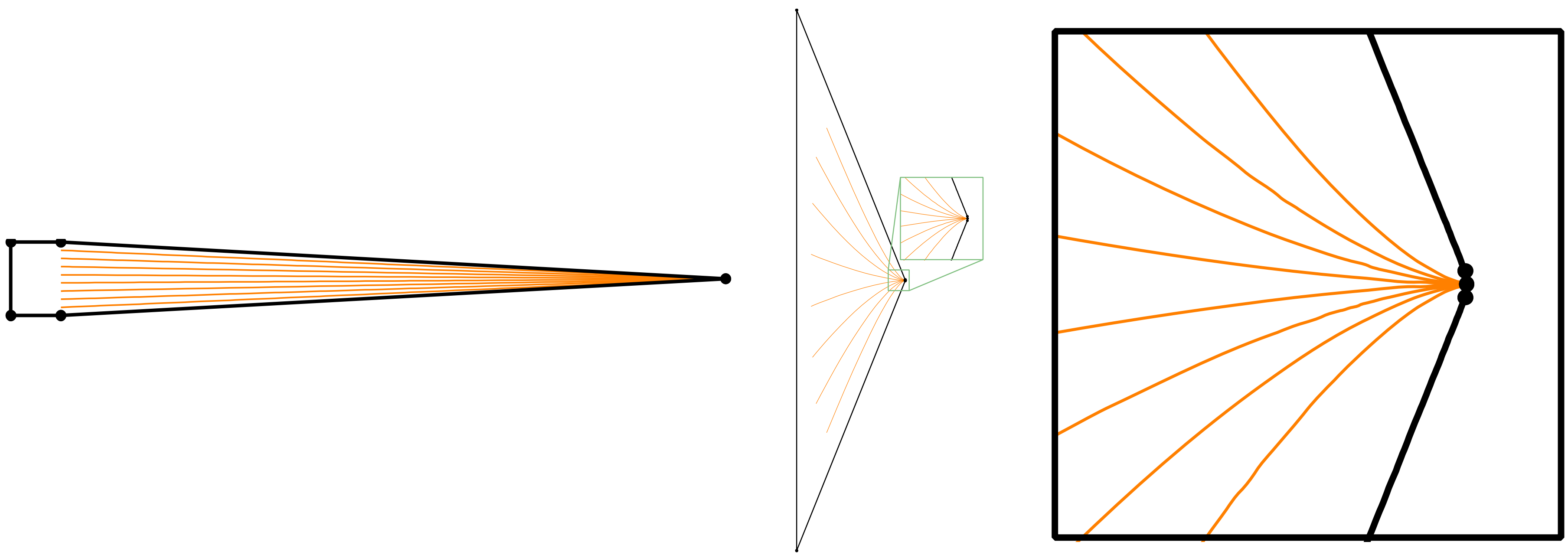


$$\Omega^0 = [v_1^0, \dots, v_n^0]$$

$$\Omega^1 = [v_1^1, \dots, v_n^1]$$

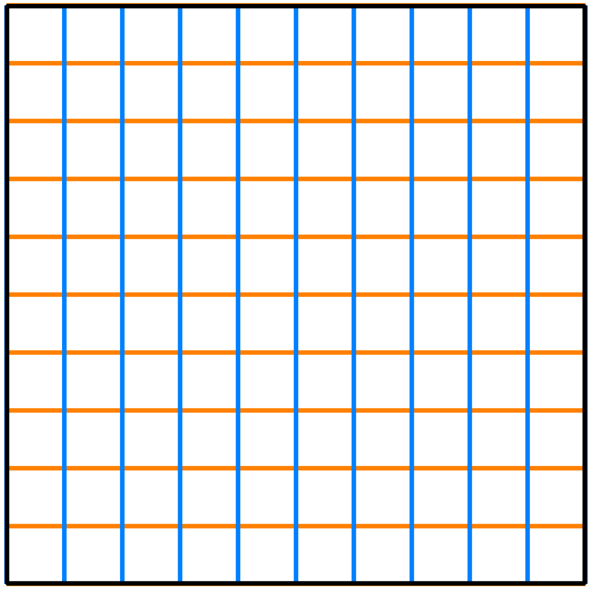
Wachspress mappings

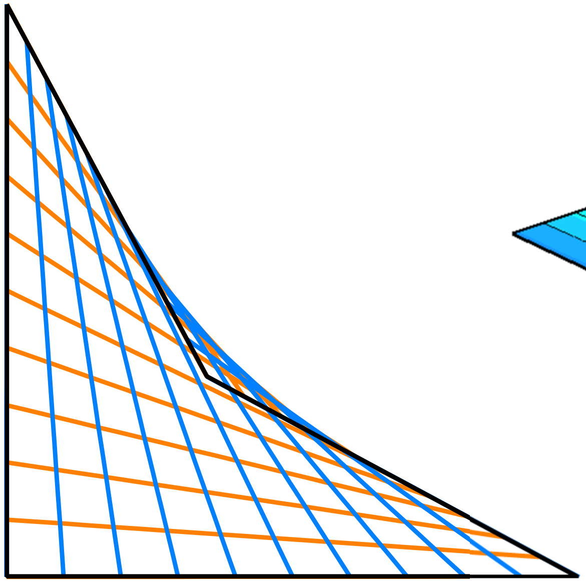
- based on WP coordinates [Wachspress 1975]
- bijective for convex polygons [Floater & Kosinka 2010]

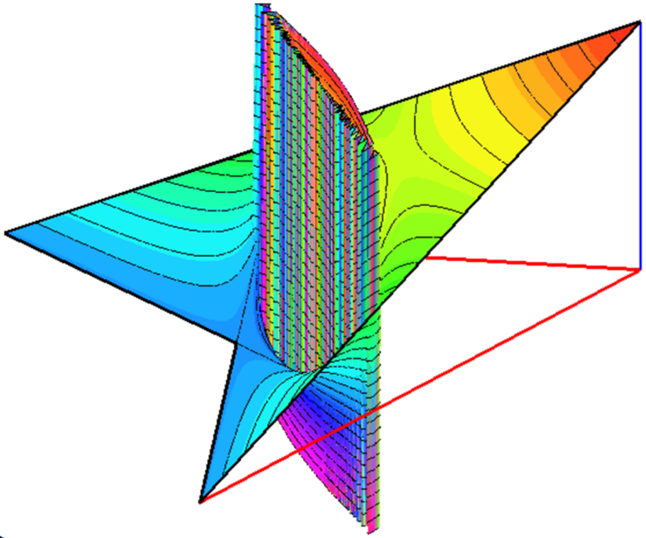


Wachspress mappings

- based on WP coordinates [Wachspress 1975]
- bijective for convex polygons [Floater & Kosinka 2010]
- not bijective for non-convex target
- not well-defined for non-convex source

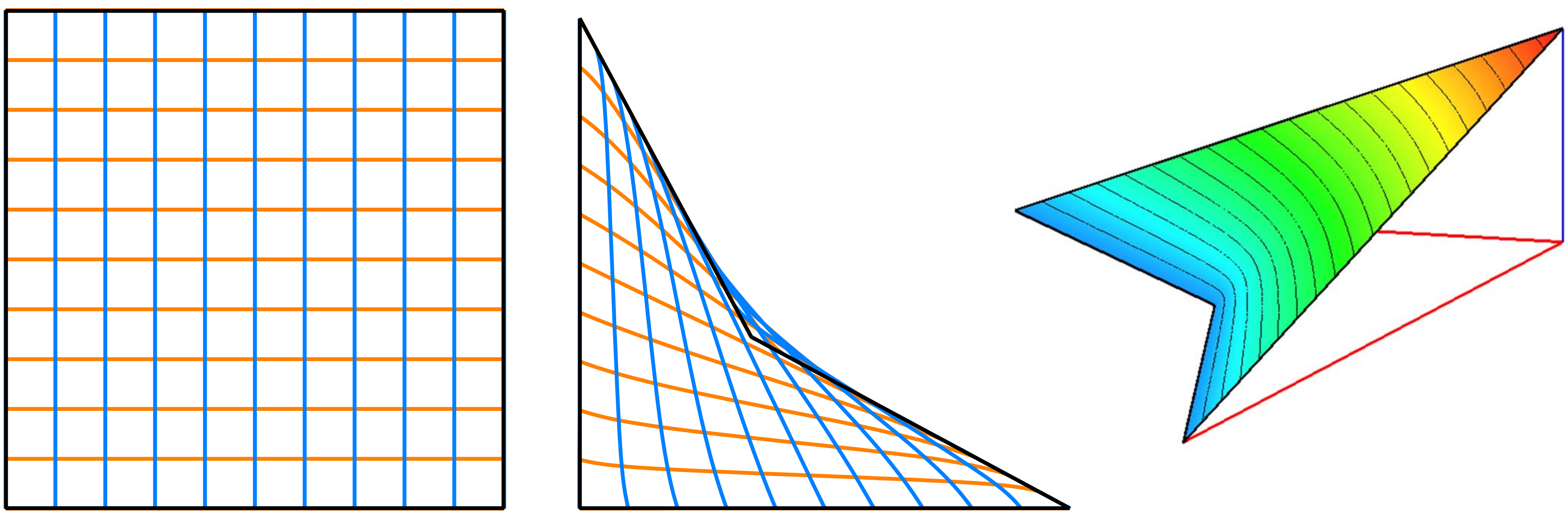






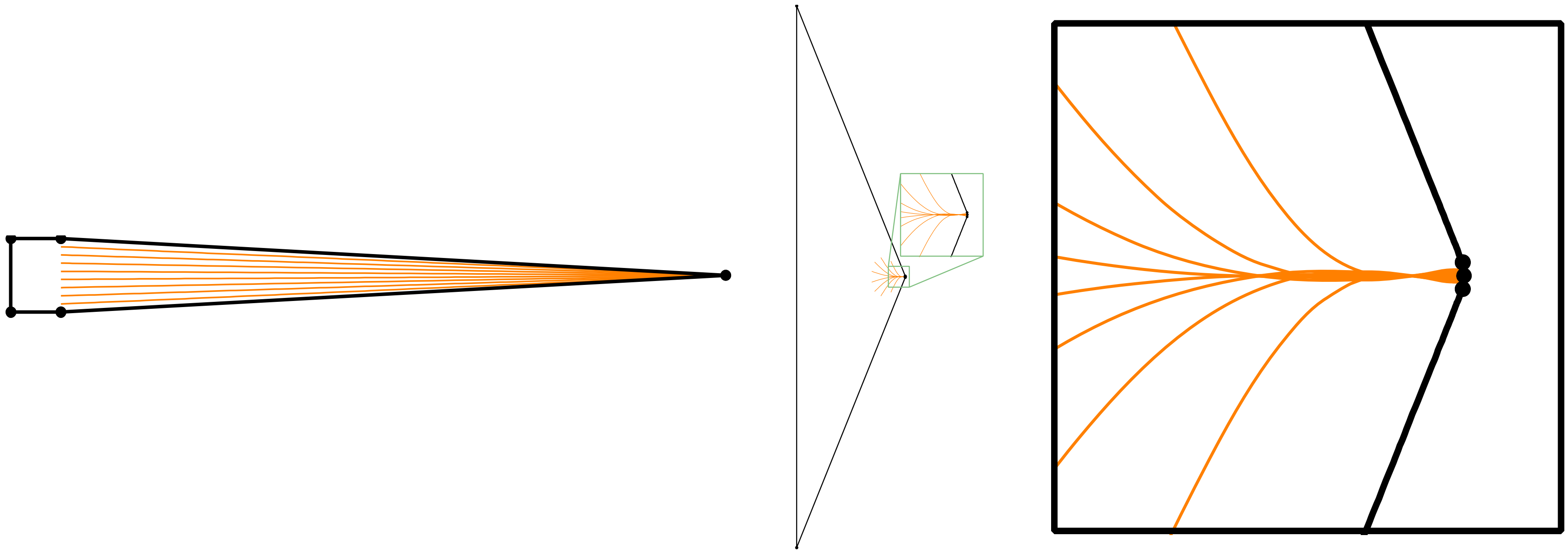
Mean value mappings

- based on MV coordinates [Floater 2003]
- well-defined for non-convex source
- not bijective



Mean value mappings

- based on MV coordinates [Floater 2003]
- well-defined for non-convex source
- not bijective, even for convex polygons

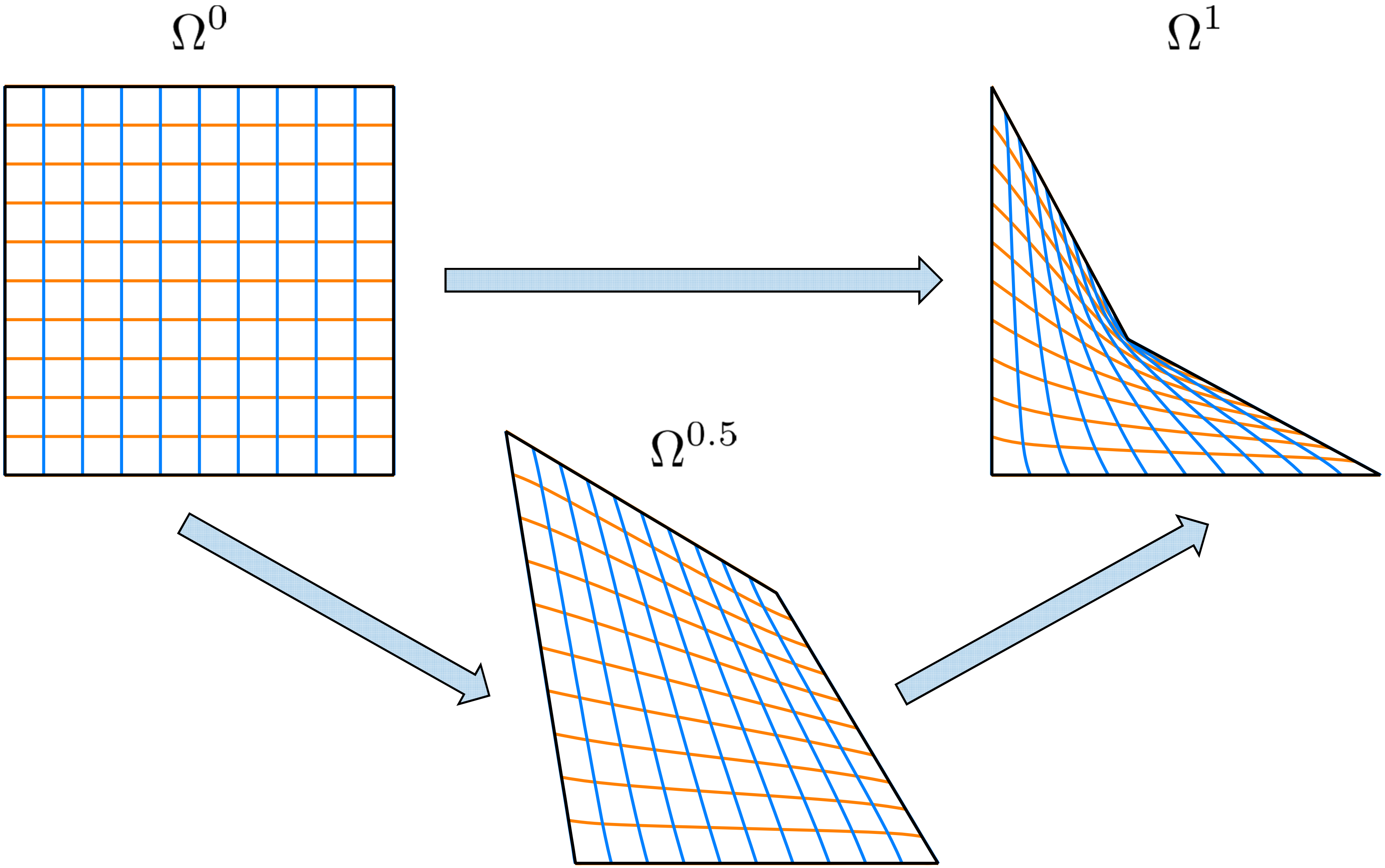


Barycentric mappings

	WP	MV
convex $\to$ convex	✓	✓
convex $\to$ non-convex	✓	✓
non-convex $\to$ convex	✗	✓
non-convex $\to$ non-convex	✗	✓

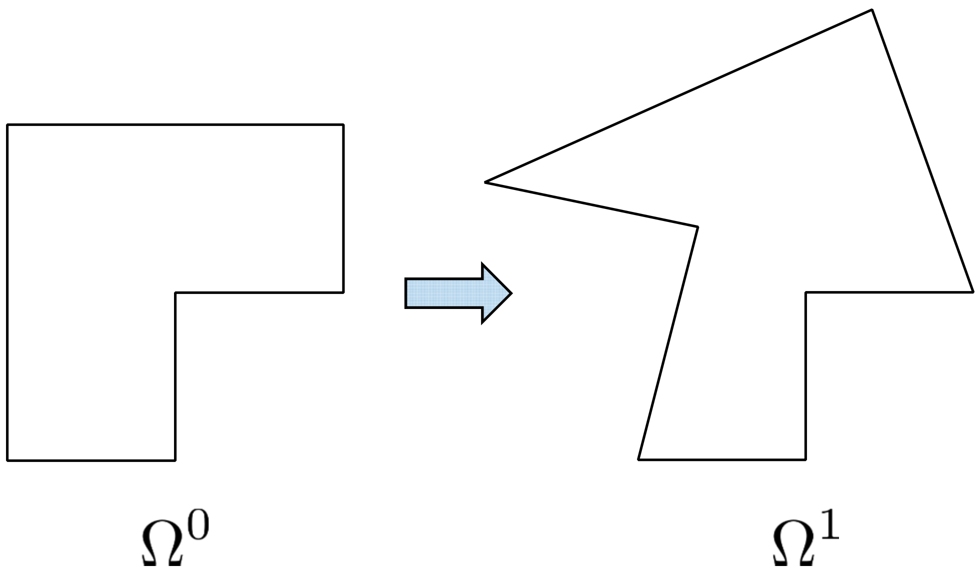
- general barycentric mappings [Jacobson 2012]
  - not bijective for non-convex target polygon

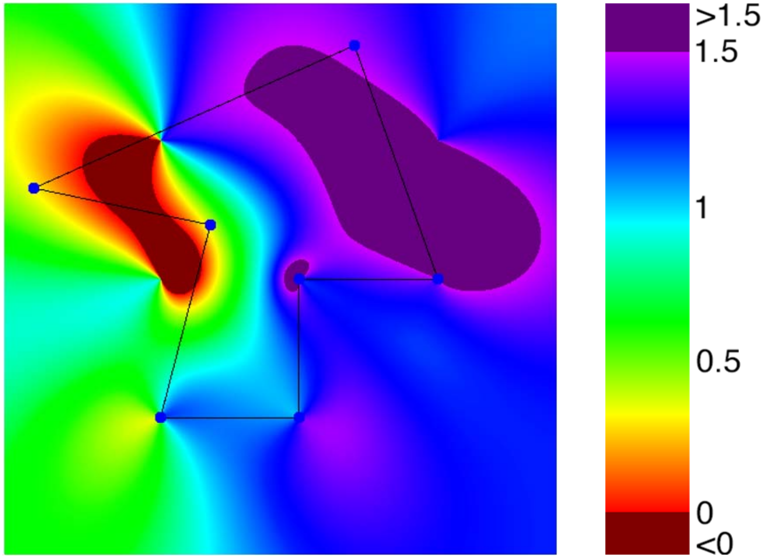
Composite barycentric mappings



Sufficient condition

- $f : \Omega^0 \to \Omega^1$ is bijective, if [Meisters & Olech 1963]
  - Ω^0 and Ω^1 without self-intersection
  - f bijective on the boundary
  - $J_f = \begin{vmatrix} \partial_1 f^1 & \partial_2 f^1 \\ \partial_1 f^2 & \partial_2 f^2 \end{vmatrix} > 0$





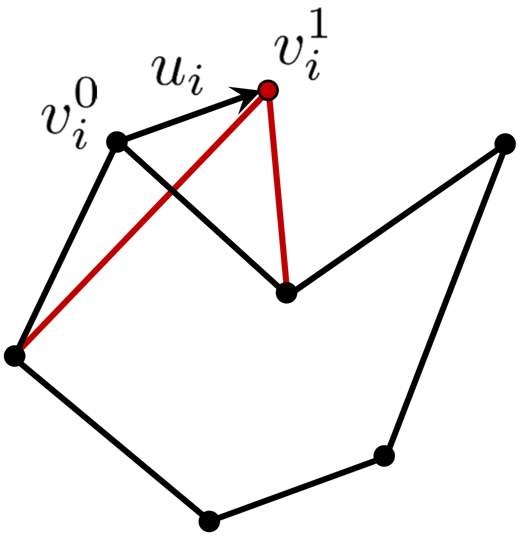
Perturbation bounds

- move one v_i^0 to $v_i^1 = v_i^0 + u_i$
  - f bijective, if

$$\|u_i\| < \frac{1}{M_i}$$

with

$$M_i = \sup_{v \in \Omega^0} \|\nabla b_i(v)\|$$

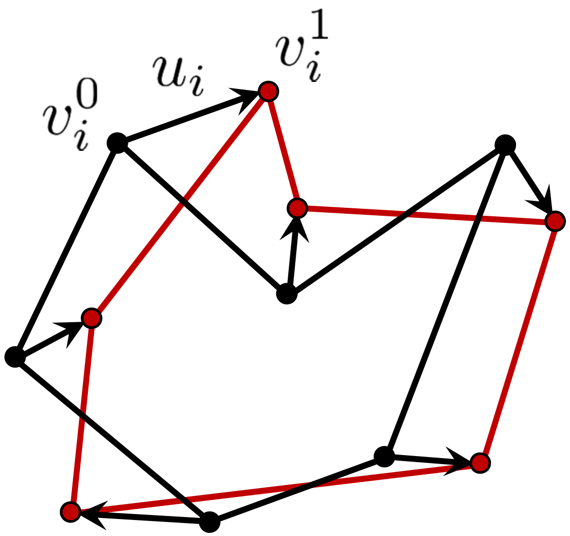


- move all v_i^0 to $v_i^1 = v_i^0 + u_i$
  - f bijective, if

$$\max_{1 \le i \le n} \|u_i\| < \frac{\sqrt{5} - 1}{2M}$$

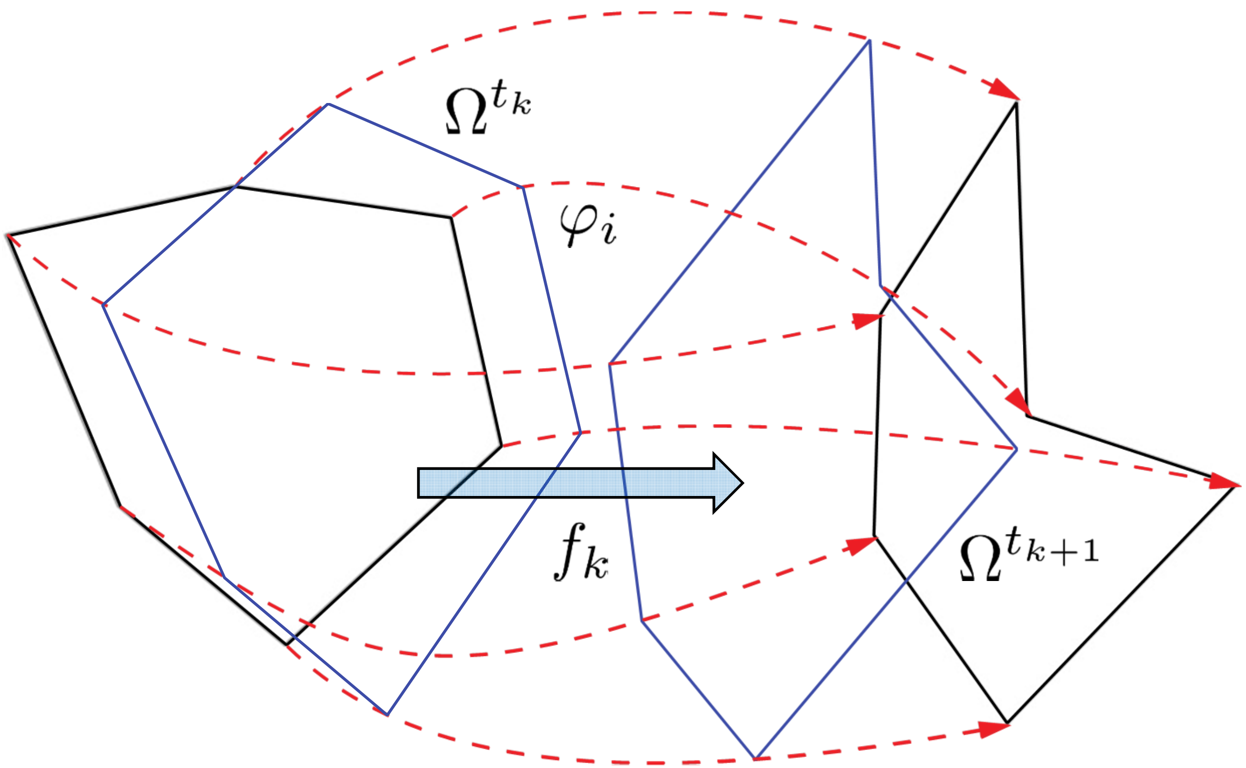
with

$$M = M_1 + \cdots + M_n$$



Composite barycentric mappings

- continuous vertex paths $\varphi_i\colon [0,1] \to \mathbb{R}^2$
- intermediate polygons $\Omega^{t_k} = [\varphi_1(t_k), \dots, \varphi_n(t_k)]$
- barycentric mappings $f_k\colon \Omega^{t_k} \to \Omega^{t_{k+1}}$



Composite barycentric mappings

- partition $\tau = (t_0, t_1, \dots, t_m)$ of $[0,1]$
- composite barycentric mapping

$$f_\tau = f_{m-1} \circ f_{m-2} \circ \cdots \circ f_1 \circ f_0$$

- f_τ bijective, if

max displacement

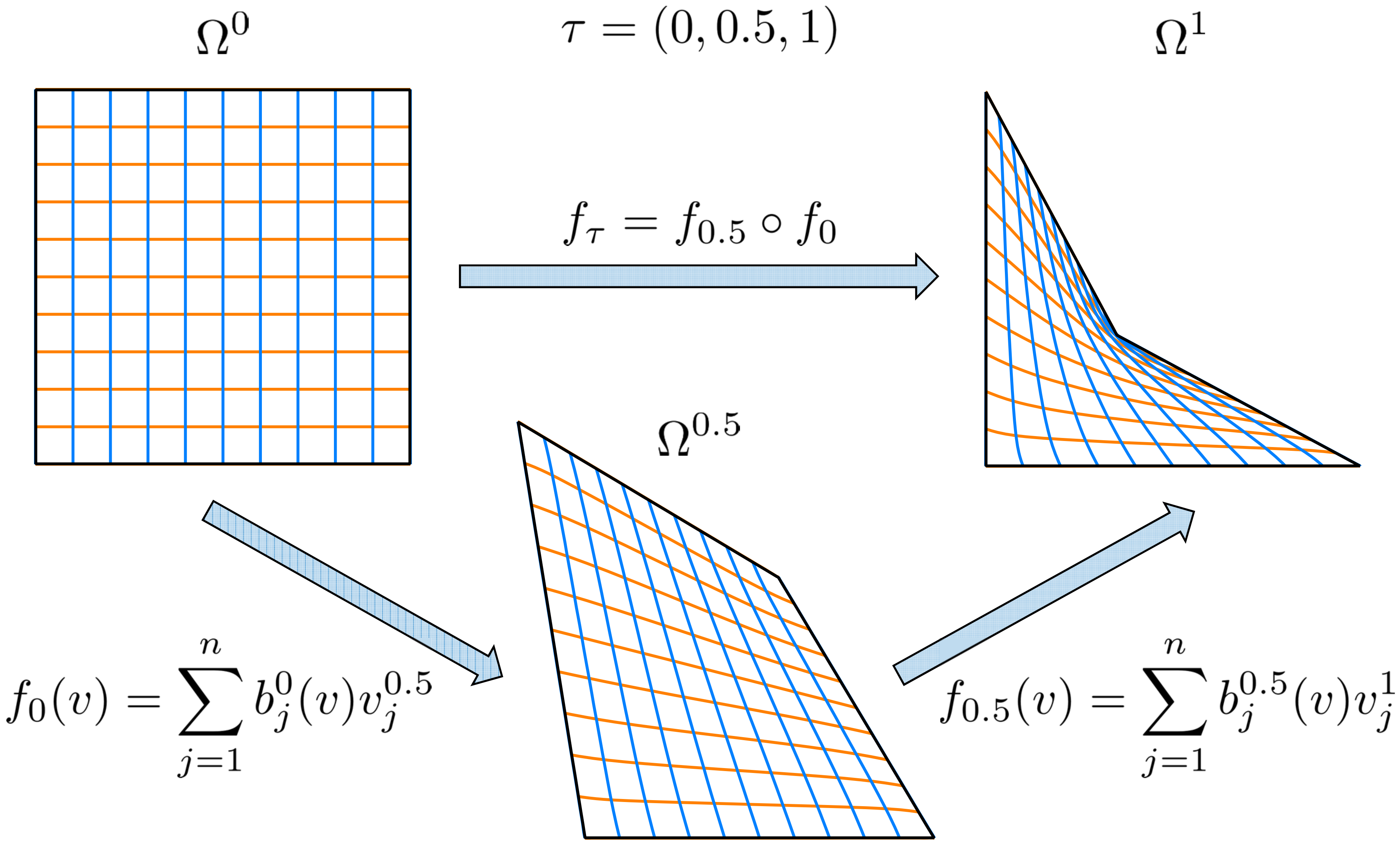
$$\max_{0 \le k < m} \max_{1 \le i \le n} \| v_i^{t_k} - v_i^{t_{k+1}} \| < \frac{\sqrt{5}-1}{2nM^*}$$

with

$$M^* = \max_{1 \le i \le n} \max_{t \in [0,1]} \sup_{v \in \Omega^t} \| \nabla b_i^t(v) \|$$

max gradient

Composite barycentric mappings

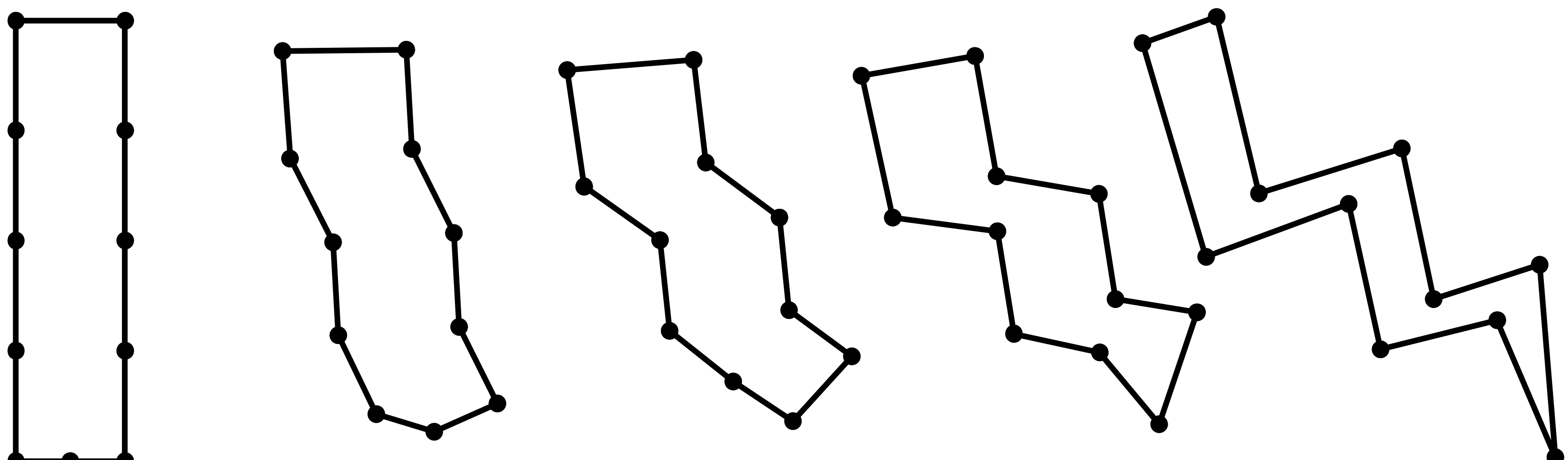


Composite mean value mappings

- use mean value coordinates to define mappings f_k
  - well-defined, as long as Ω^{t_k} without self-intersections
- $\|\nabla b_i^t(v)\|$ bounded for $v \in \Omega^t$
  - if Ω^t is convex [Rand et al. 2012]
  - if Ω^t is non-convex $\Rightarrow$ future work
- constant M^* is finite
- f_τ bijective for uniform steps $t_i = i/m$
  - continuous vertex paths φ_i
  - m sufficiently large

Vertex paths

- Ω^t by linearly interpolating [Sederberg at al. 1993]
  - edges lengths
  - signed turning angles
  - barycentre
  - orientation of one edge



$t=0$ $t=0.25$ $t=0.5$ $t=0.75$ $t=1$

Adaptive binary partition

checkInterval (0, 1)

J_{min} = computeJmin (,)

if $J_{min} \leq 0$ then

$c = (\quad + \quad)/2$

$\tau = \tau \cup c$

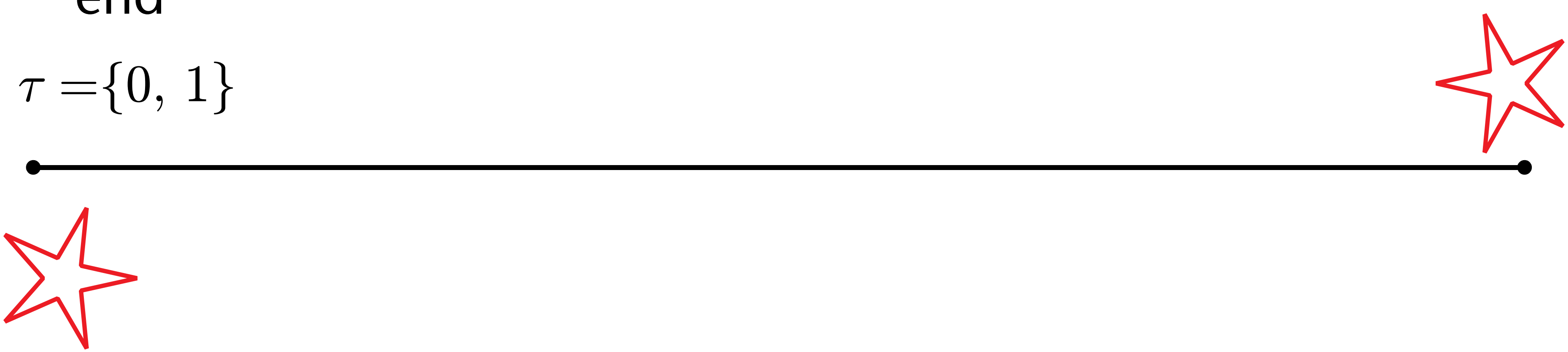
checkInterval (, c)

checkInterval (c,)

end

end

$\tau = \{0,\ 1\}$



Adaptive binary partition

checkInterval $(0, 1)$

$J_{min} =$ computeJmin $(0, 1)$

if $J_{min} \leq 0$ then

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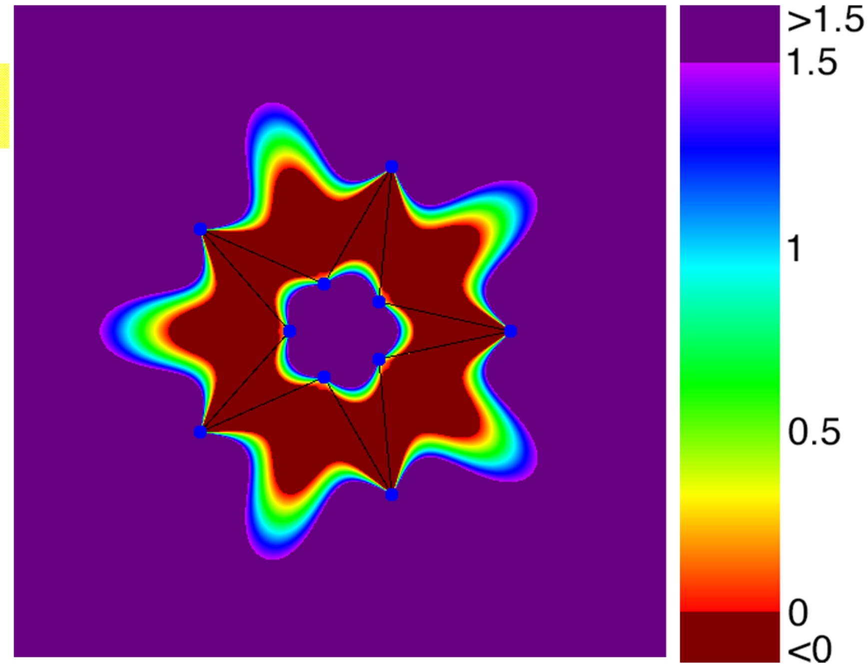
$\tau = \tau \cup c$

checkInterval $(\quad, c)$

checkInterval $(c, \quad)$

end

end



$\tau = \{0, 1\}$

Adaptive binary partition

checkInterval $(0, 1)$

J_{min} = computeJmin $(0, 1)$

if $J_{min} \leq 0$ then

$c = (0 + 1)/2$

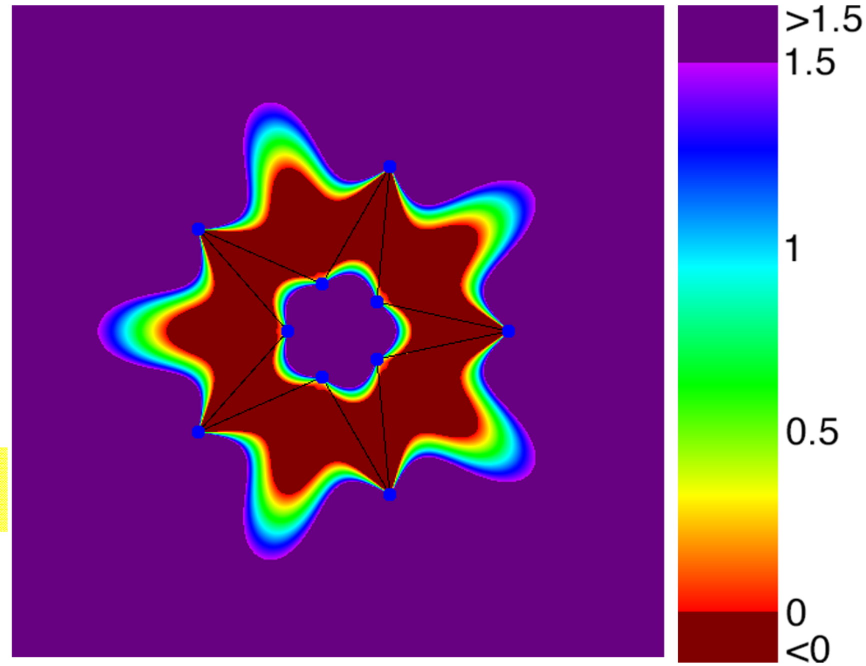
$\tau = \tau \cup c$

checkInterval $(0, c)$

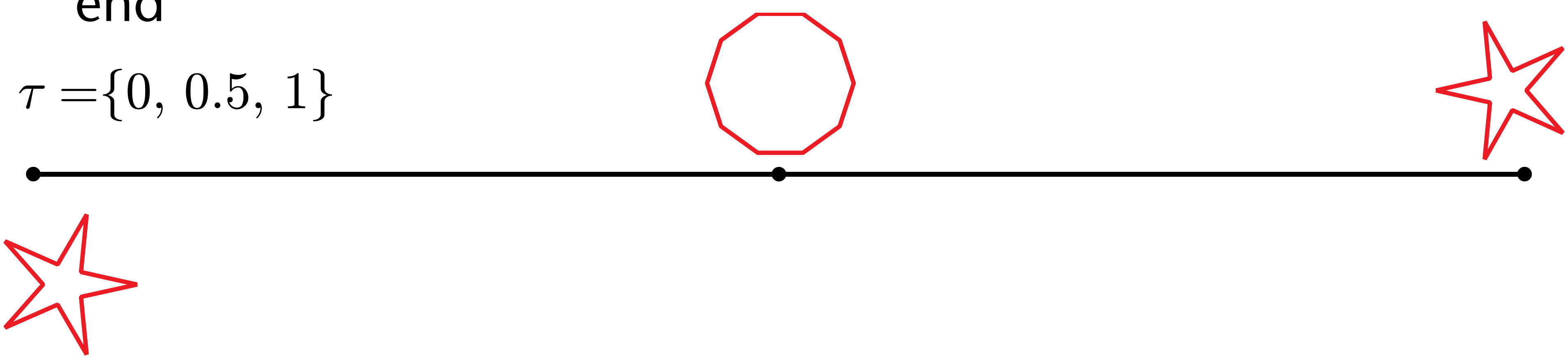
checkInterval $(c, 1)$

end

end



$\tau = \{0,\ 0.5,\ 1\}$



Adaptive binary partition

checkInterval $(0, 0.5)$

➡ $J_{min} =$ computeJmin $(0, 0.5)$

if $J_{min} \leq 0$ then

$c = (\quad + \quad)/2$

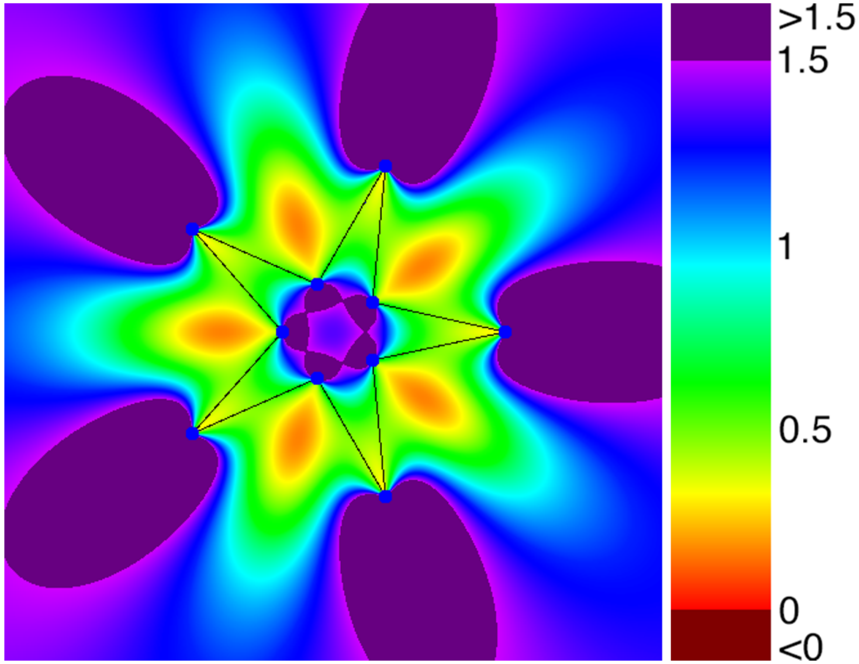
$\tau = \tau \cup c$

checkInterval $(\quad, c)$

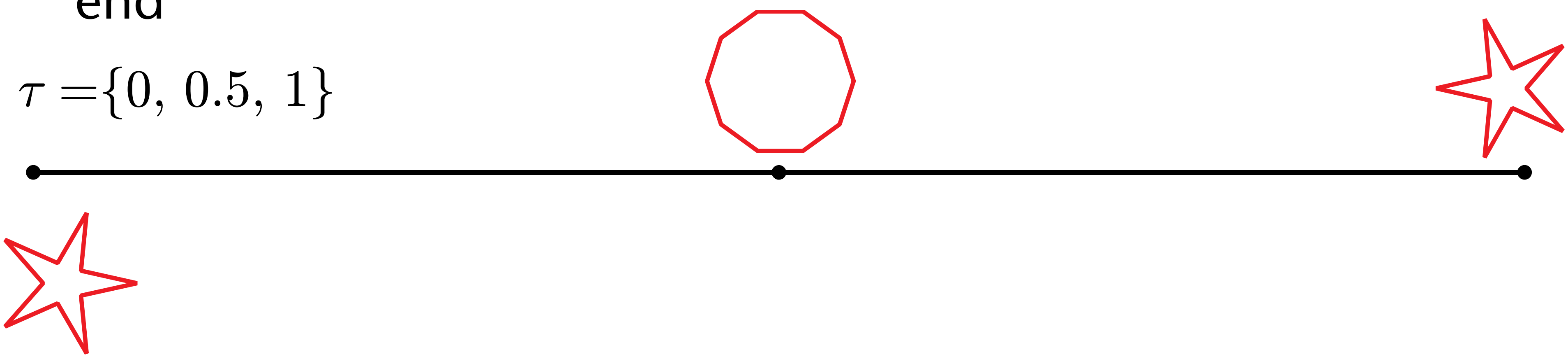
checkInterval $(c, \quad)$

end

end



$\tau = \{0,\ 0.5,\ 1\}$



Adaptive binary partition

checkInterval $(0, 0.5)$

J_{min} = computeJmin $(0, 0.5)$

if $J_{min} \leq 0$ then

$c = (\quad + \quad)/2$

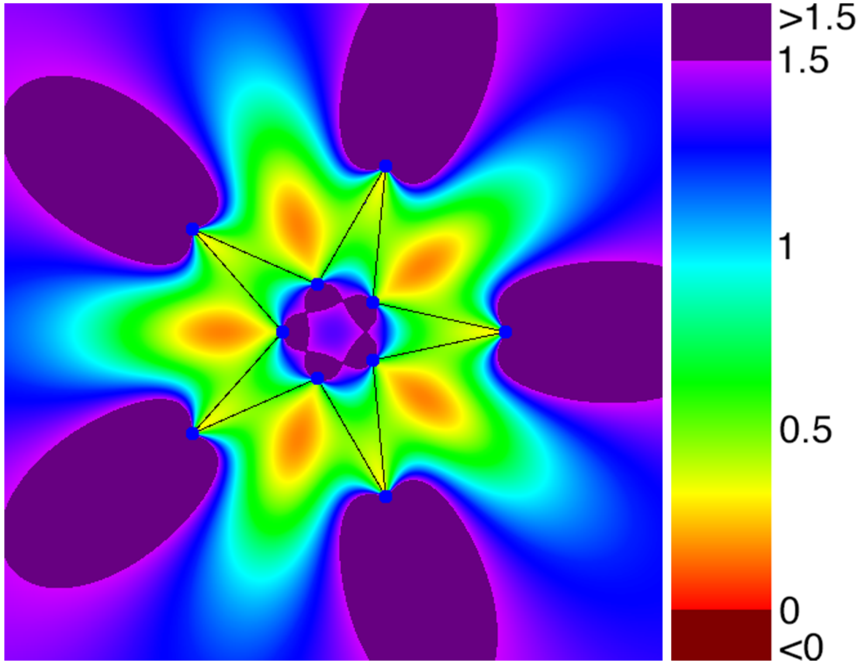
$\mathcal{T} = \mathcal{T} \cup c$

checkInterval $(\quad , c)$

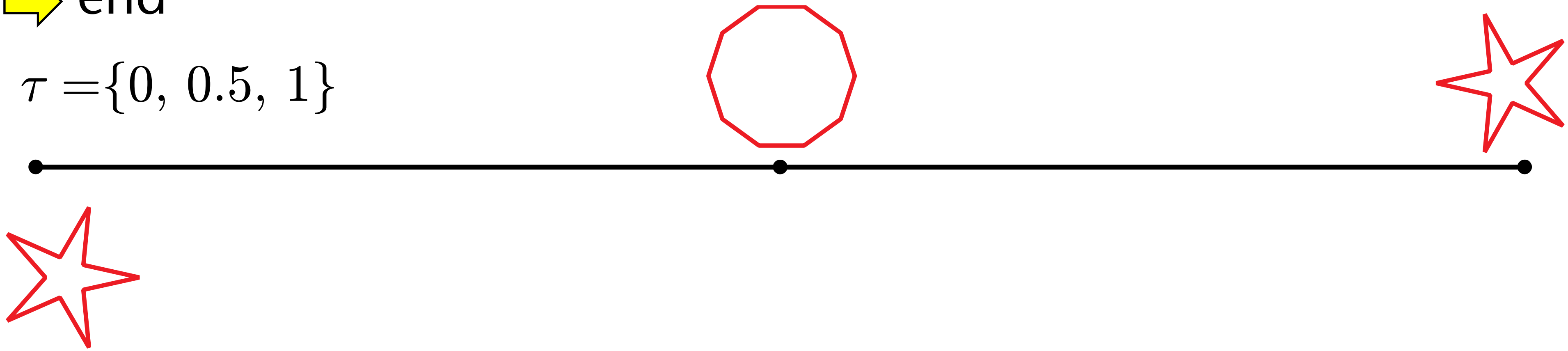
checkInterval $(c, \quad)$

end

end



$\mathcal{T} = \{0,\ 0.5,\ 1\}$



Adaptive binary partition

checkInterval $(0.5, 1)$

➡ J_{min} = computeJmin $(0.5, 1)$

if $J_{min} \leq 0$ then

$c = (\quad + \quad)/2$

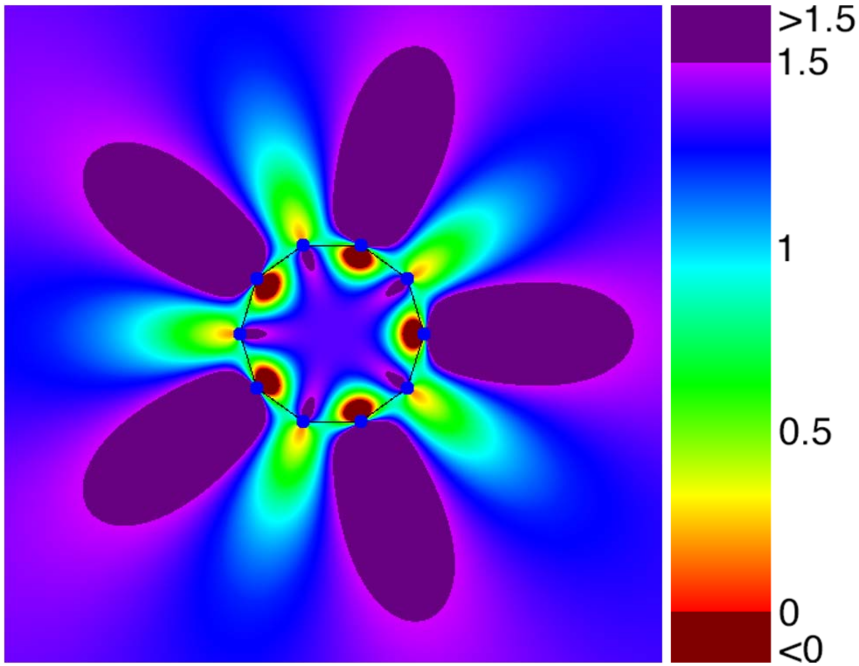
$\tau = \tau \cup c$

checkInterval $(\quad , c)$

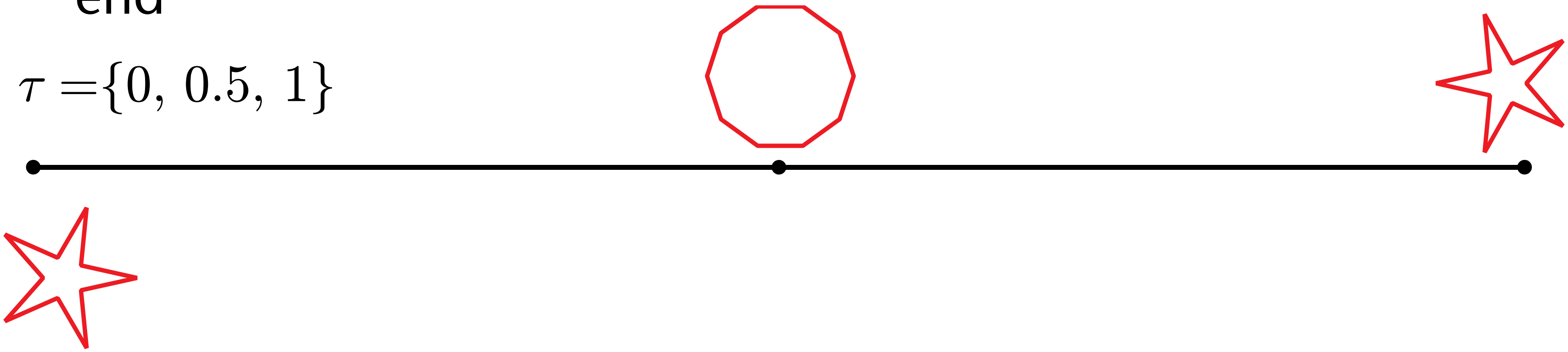
checkInterval $(c, \quad)$

end

end



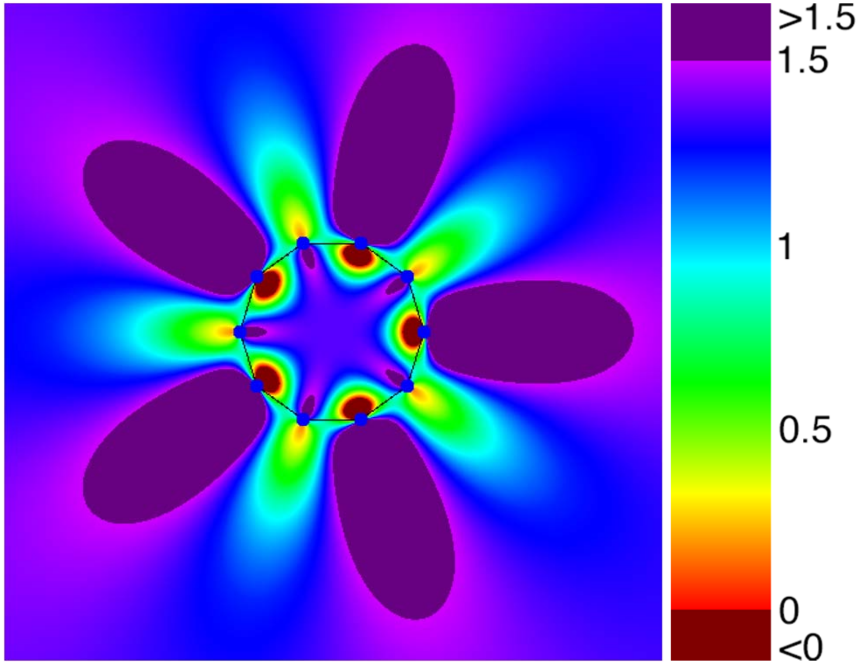
$\tau = \{0,\ 0.5,\ 1\}$



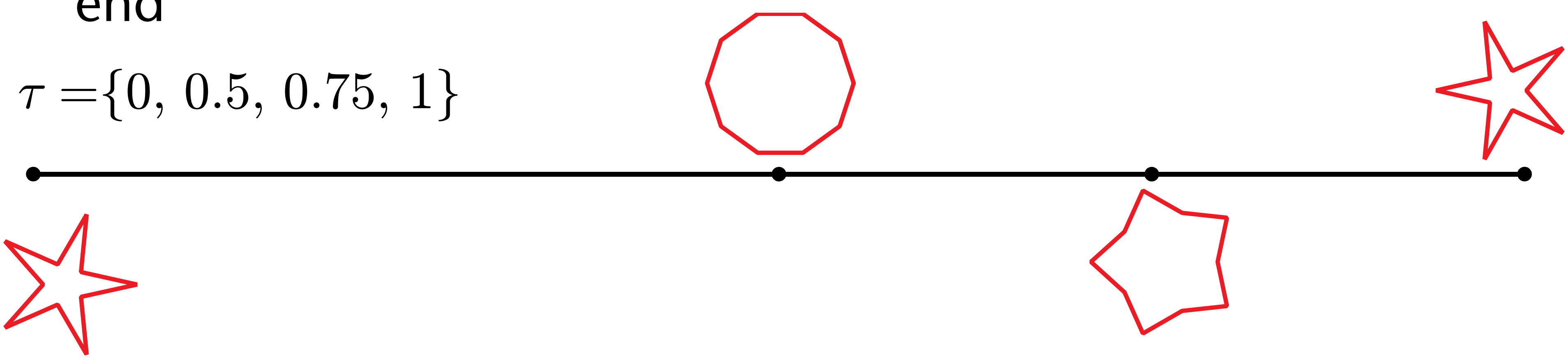
Adaptive binary partition

checkInterval $(0.5, 1)$
J_{min} = computeJmin $(0.5, 1)$

if $J_{min} \leq 0$ then
$c = (0.5 + 1)/2$
$\tau = \tau \cup c$
checkInterval $(0.5, c)$
checkInterval $(c, 1)$
end
end



$\tau = \{0,\ 0.5,\ 0.75,\ 1\}$



Adaptive binary partition

checkInterval $(0.5, 1)$

$J_{min} =$ computeJmin $(0.5, 1)$

if $J_{min} \leq 0$ then

$c = (0.5 + 1)/2$

$\tau = \tau \cup c$

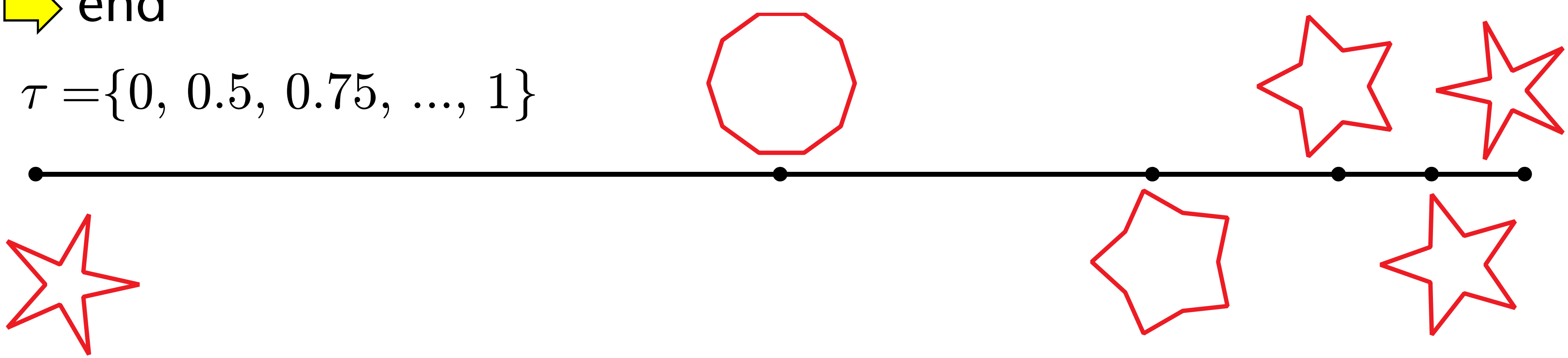
checkInterval $(0.5, c)$

checkInterval $(c, 1)$

end

end

$\tau = \{0, 0.5, 0.75, ..., 1\}$



Composite barycentric mapping

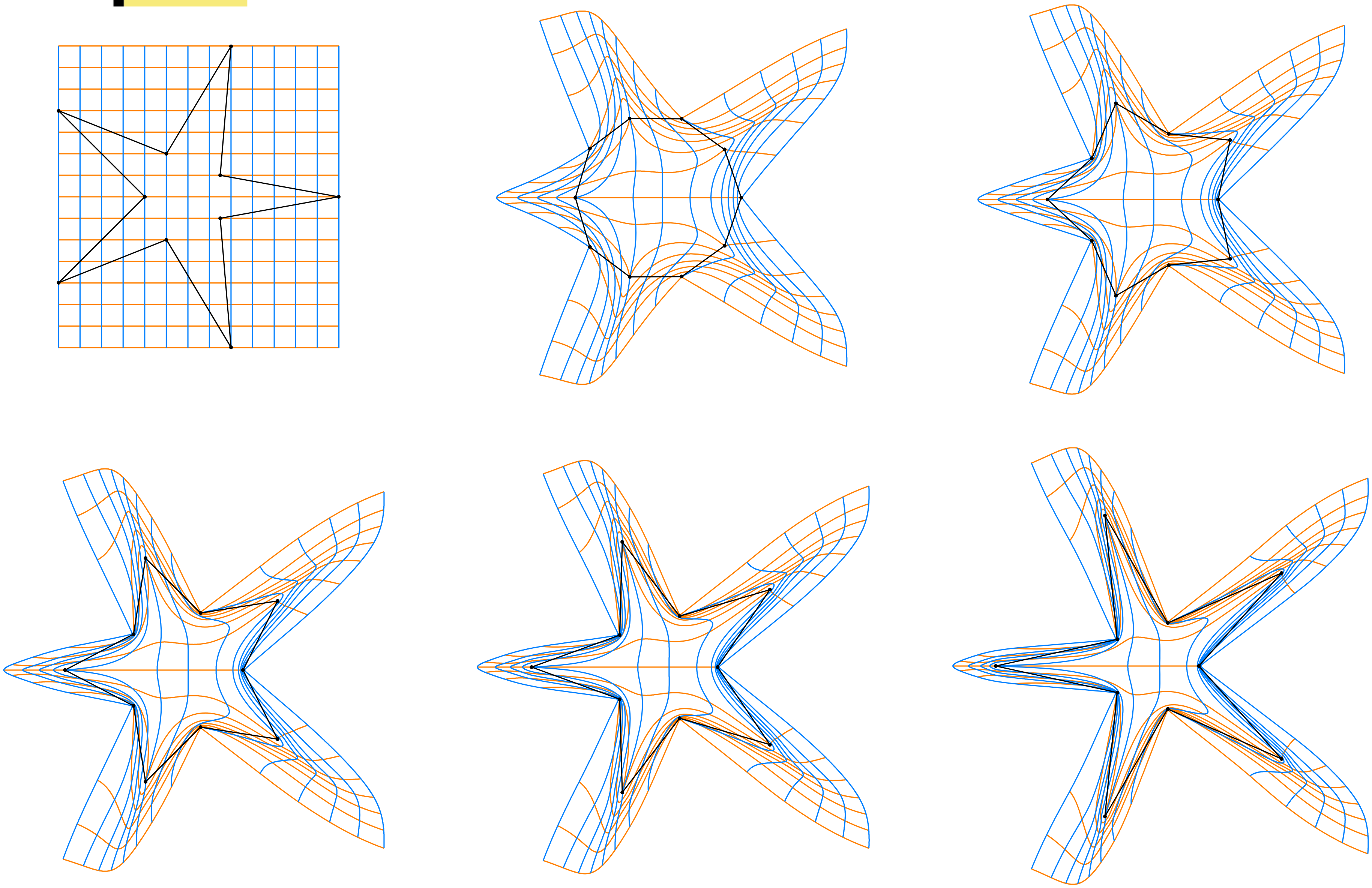
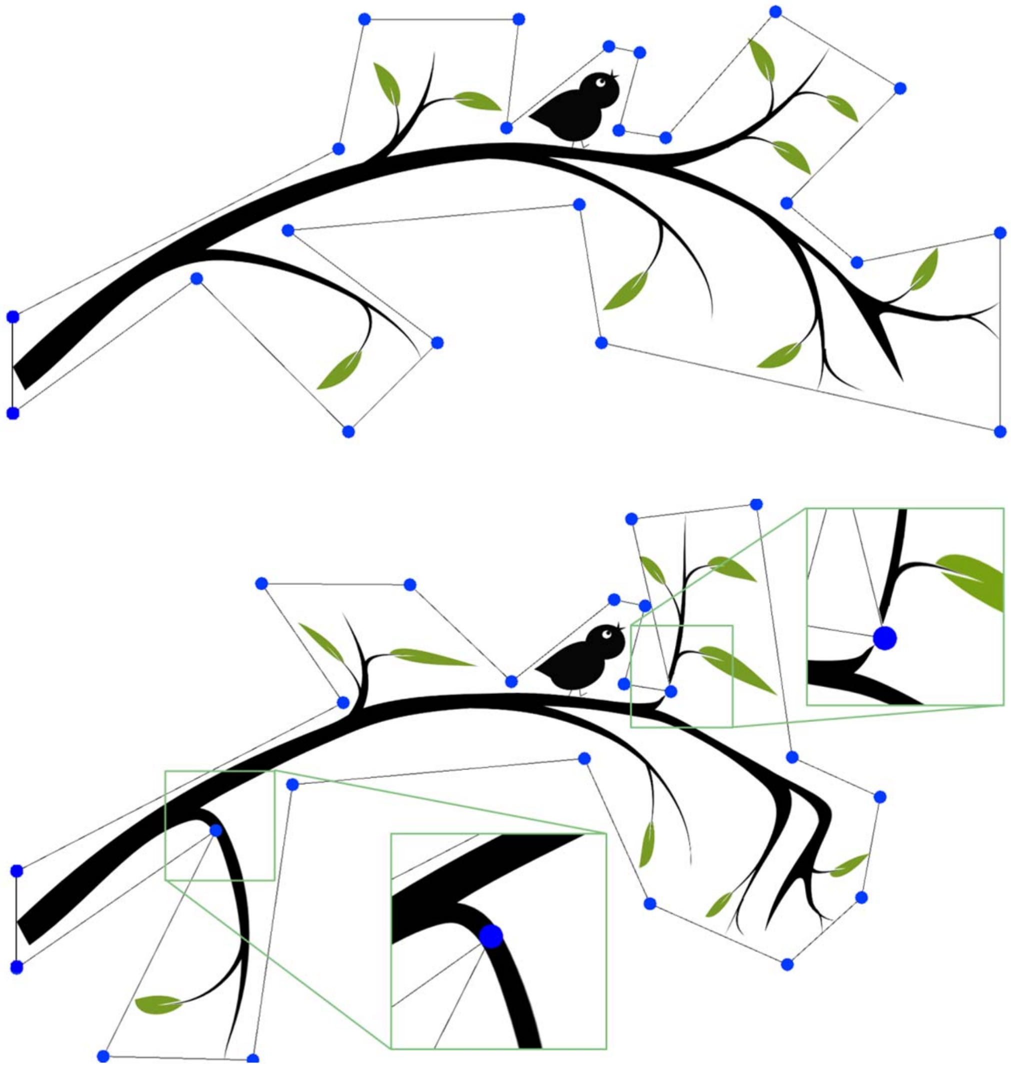
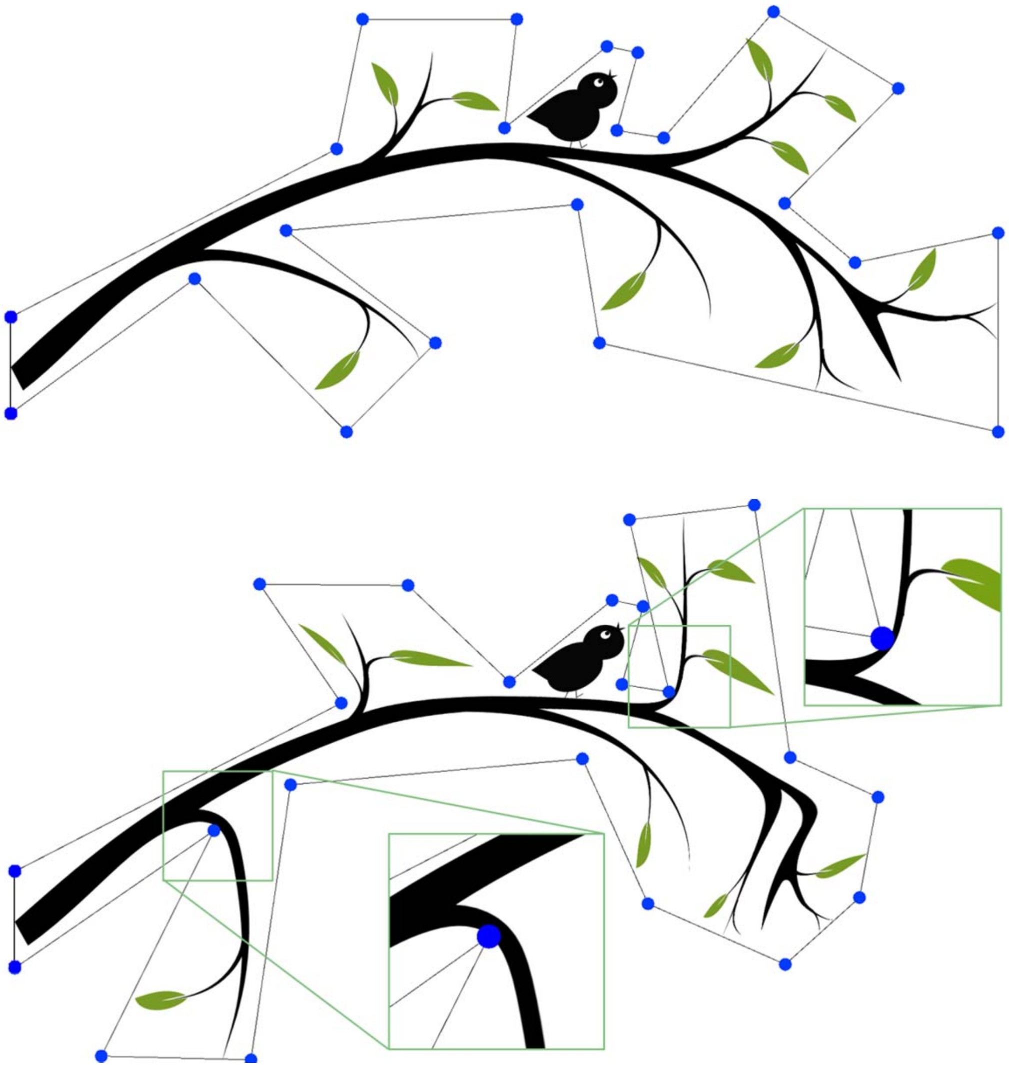


Image warping comparison



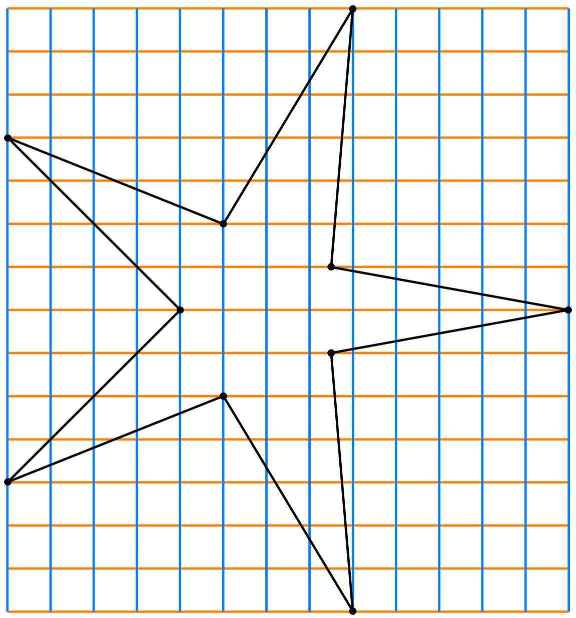
mean value

Image warping comparison

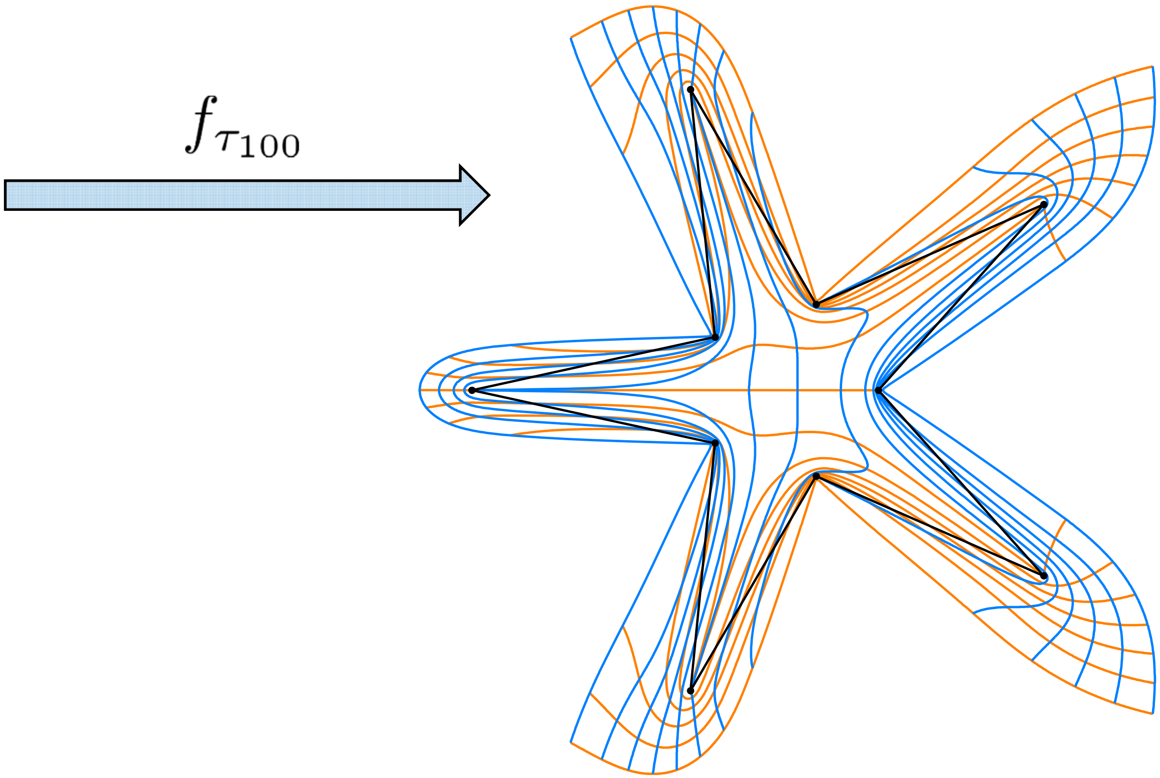


composite mean value

Infinite barycentric mappings

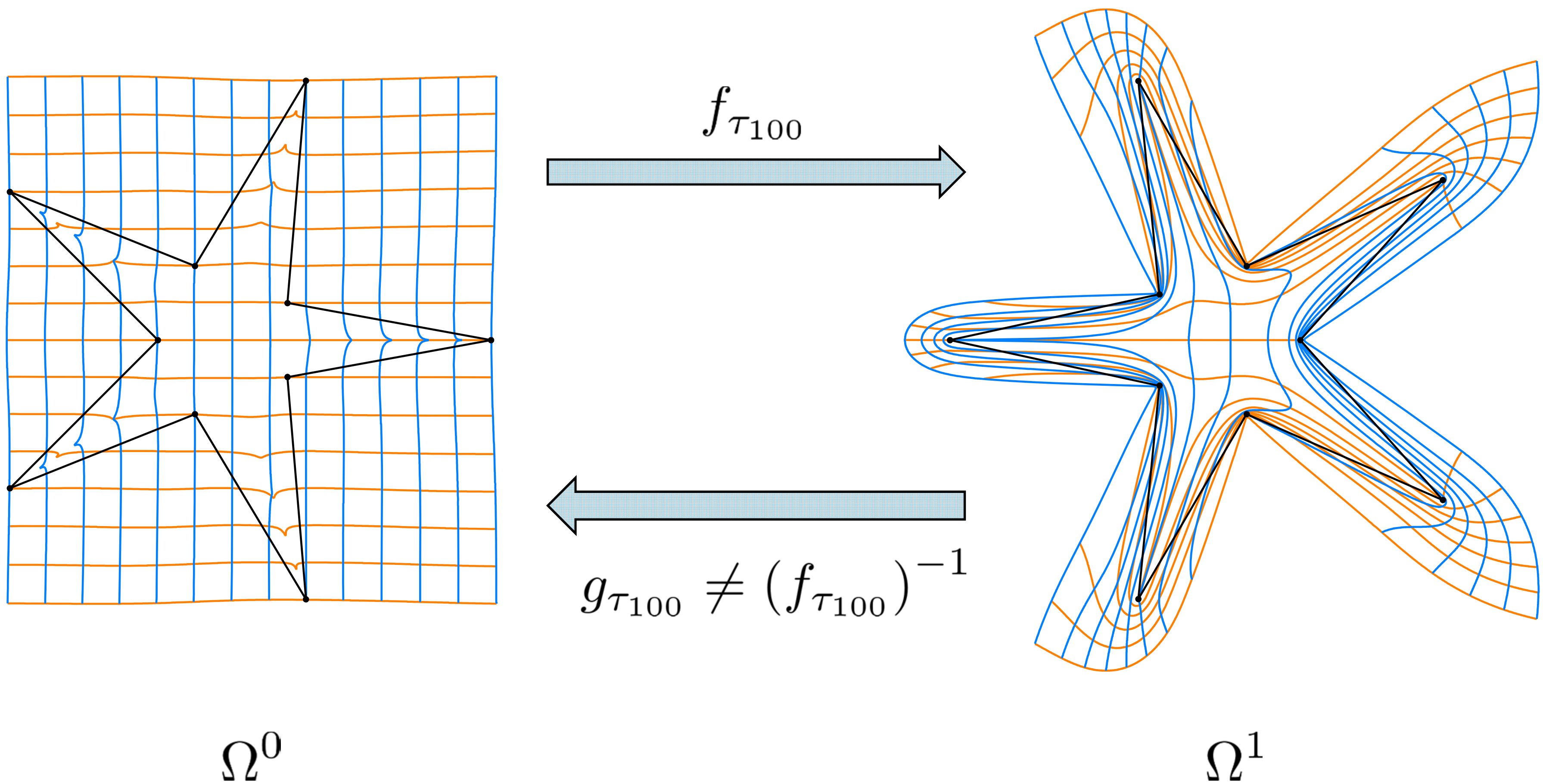


$f_{\tau_{100}}$



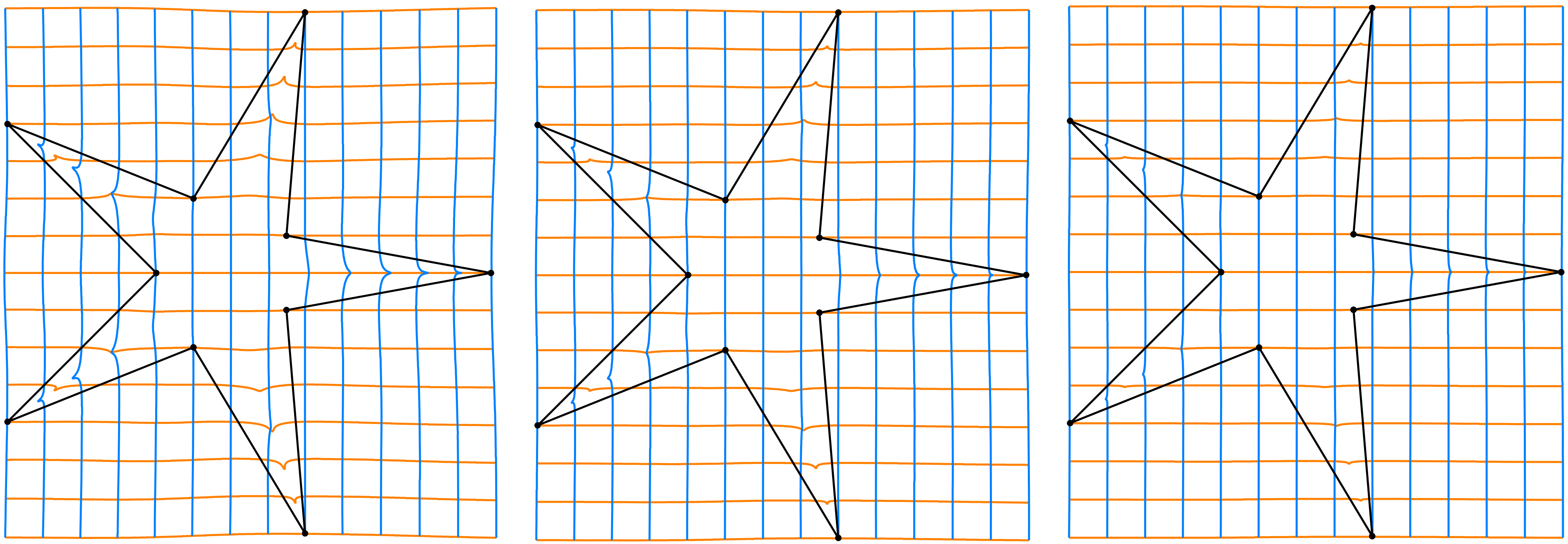
Ω^0 Ω^1

Infinite barycentric mappings



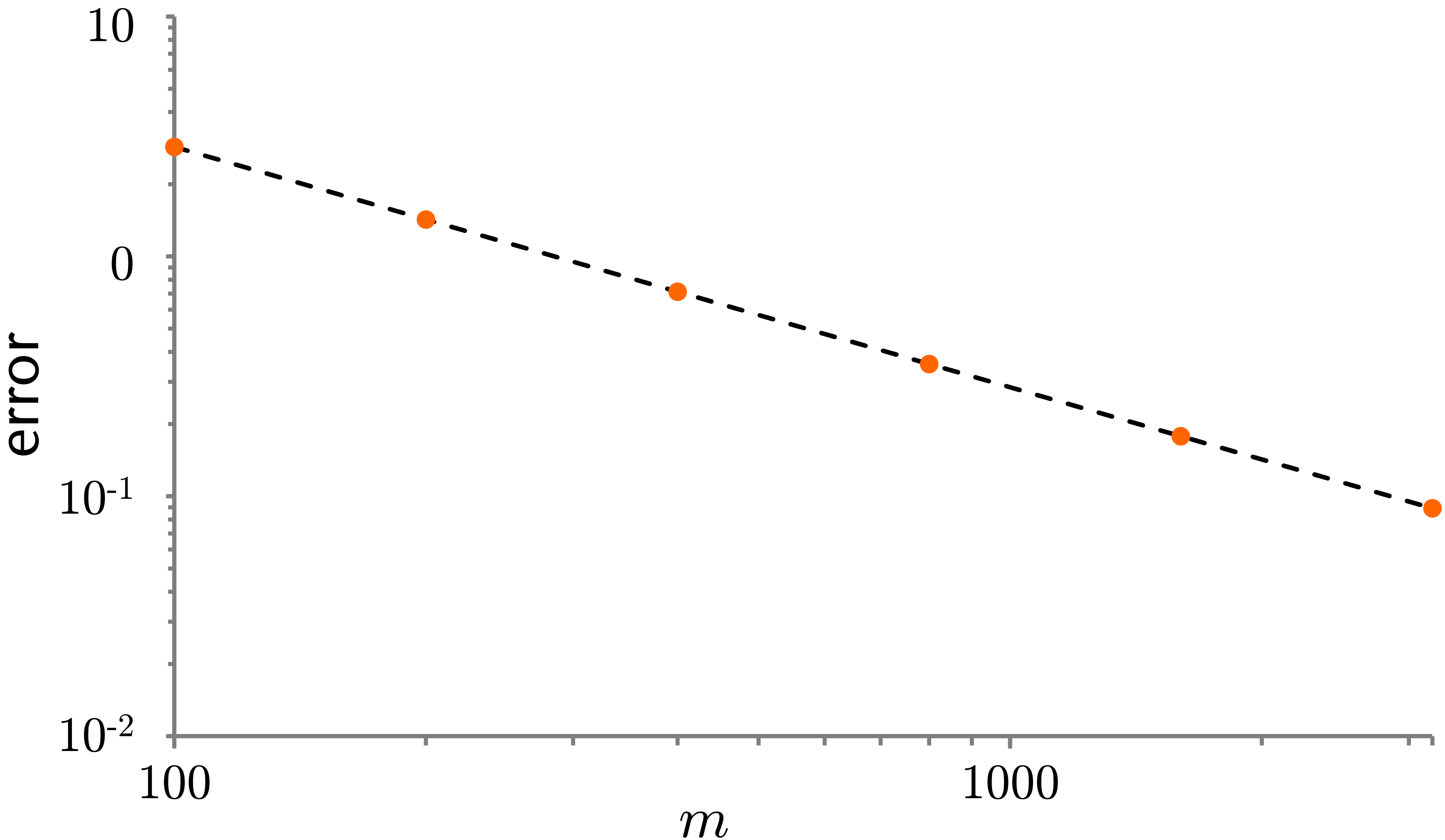
Infinite barycentric mappings

$$g_{\tau_m} \circ f_{\tau_m}$$



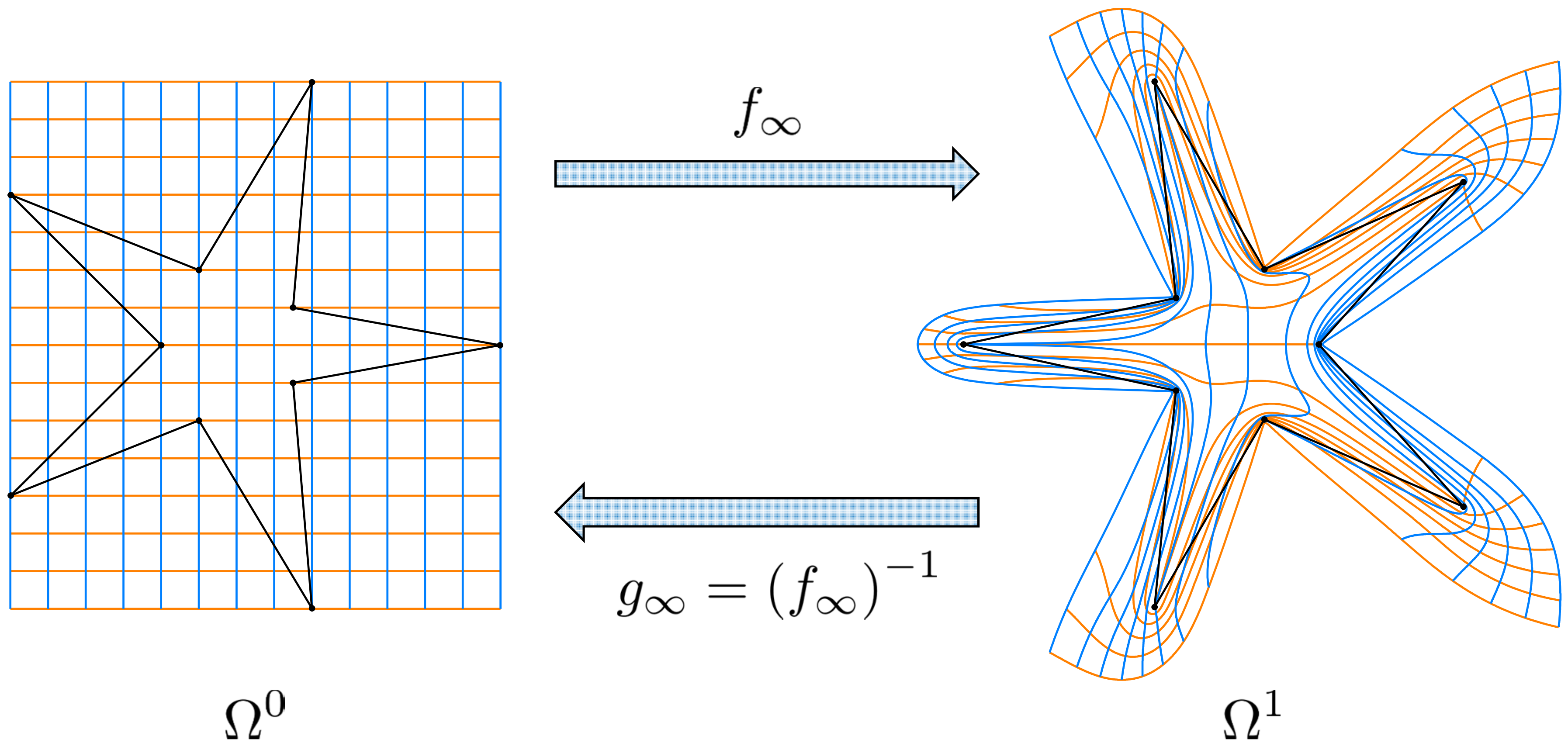
$m = 100$ $m = 200$ $m = 400$

Infinite barycentric mappings



Infinite barycentric mappings

- infinite barycentric mapping: $f_\infty = \lim_{m\to\infty} f_{\tau_m}$



Conclusion

- construction of bijective barycentric mappings
  - composition of intermediate mappings
  - theoretical bounds on the displacement
- real-time composite mean value mappings
  - construction of the adaptive binary partition
  - real-time GPU implementation
- infinite composite mappings
  - natural inverse
  - empiric result of convergence