Recovering surfaces with discontinuity curves from gridded data

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Aim

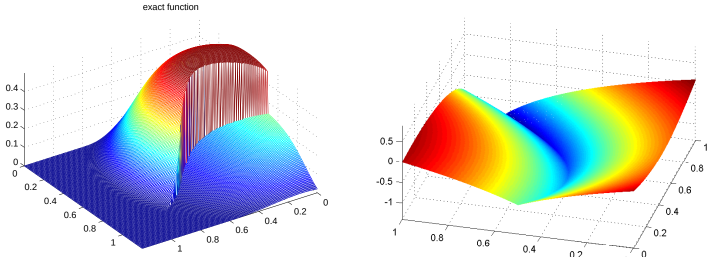
Provide a faithful recovery of surfaces presenting discontinuities when a set of gridded data is given.

Namely, we want to recover functions

$$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$$

with *vertical faults* or *oblique faults*.

- *Vertical faults*: the function f is discontinuous across a curve $\Gamma \subset \Omega$;
- *oblique faults*: the gradient of f, ∇f is discontinuous across a curve $\Gamma \subset \Omega$.



Motivations

Surfaces with discontinuities appear in many scientific applications including: signal and image processing, geophysics....

- Analysis of medical images as the magnetic resonance (MRI). *Vertical faults* may indicate the presence of some pathology.
- *Vertical* and *oblique* occur in many problems of geophysical interest when describing the shape of geological entities as
  - the topography of seafloor surfaces,
  - mountainous districts.

Discretely defined surfaces that exhibit such features cannot be correctly recovered without the knowledge of

- the position of the discontinuity curves Γ
- the type of discontinuity.
- a good recovery of the discontinuity curve Γ

Otherwise, typical problems that occur are

- undue oscillations
- poor approximation near gradient faults.

Detection

- Wide literature related to image analysis concerning vertical fault (edge) detection when data are placed on a uniform grid and the large sample size N is at least 2^{16}. Recent papers in this area include [Arandiga et al. 2008], [Plonka 2009], [R. 2009] and the references therein.
- For scattered locations and moderate size $N < 2^{16}$
  - Vertical fault detection: [Jung, Gottlieb, Kim 2011], [Allasia, Besenghi, De Rossi 2000], [Allasia, Besenghi, Cavoretto 2009-1], [Archibald Gelb, Yoon 2005], [Campton, Mason 2005], [Iske 1997],[Lòpez de Silanes, Parra, Torrens 2008], [R. 1998].
  - Oblique fault detection: [Lòpez de Silanes, Parra, Torrens 2004], [R. 1997], [Bozzini, R. 2013].

Approximation of Γ

Correct approximation of Γ is essential to get a faithful recovery of the surface (see e.g [Besenghi, Costanzo, De Rossi 2003], [Bozzini, R. 2000] and [Gout, Guyader, Romani 2008]).

- Only few papers giving suggestions for recovering the curve Γ, e.g. [Campton, Mason 2005], [Lòpez de Silanes, Parra, Torrens 2004];
- in [Allasia, Besenghi, Cavoretto 2009-1] and [Allasia, Besenghi, Cavoretto 2009], different methods based on polygonal line, least squares and best L_∞ approximation are proposed in order to get an accurate approximation of Γ.
- In [Bozzini, R. 2013], we show that it is not sufficient to get an accurate approximation, but it is necessary that the obtained approximation of Γ provides the same partition of the sample given by the true discontinuity curve.

Surface recovering

Few papers for the recovering, e.g

- Vertical faults: [Arge, Floater 1994], [Allasia, Besenghi, Cavoretto 2009-1] [Besenghi, Costanzo, De Rossi 2003], [Lòpez de Silanes et al. Mamern2011], [Gout, Guyader, Romani 2008],
- Oblique faults: [Bozzini, R. 2002], [Bozzini, Lenarduzzi, R. 2013]

- Here we propose an interpolation strategy which provides a faithful recovery of a faulted surfaces when gridded data are given;
- The discontinuity curve Γ is supposed known. If this were not the case, we would have first to apply a detection method and approximate Γ.

The problem

Let $f(\mathbf{x})$ be a function defined on the square domain Ω

$$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$$

- f or its gradient $\nabla f(\mathbf{x})$ are discontinuous across a curve Γ of Ω and smooth in any neighborhood $\mathcal{U}$ of Ω which does not intersect Γ.
- Γ is smooth, $y = \Gamma(x)$.
- F is a sample of gridded data of step-size h

$$F = \{(x_\beta, f(x_\beta)),\ x_\beta \in h\mathbb{Z}^2 \cap \Omega\}.$$

- Connections between either splines and Green's functions or radial basis functions and Green's functions have repeatedly been used during the past decades (see e. g. [Schumaker 1981], [Unser et al. 2005] [Fasshauer 2010]).

  Important examples are

- polyharmonic kernels

$$v_{2m-d}(r) = \begin{cases} (-1)^{\lceil m-d/2 \rceil} r^{2m-d} & 2m-d \notin 2\mathbb{Z} \\ (-1)^{1+m-d/2} r^{2m-d} \log r & 2m-d \in 2\mathbb{Z} \end{cases} \quad 2m-d > 0,$$

  which are fundamental solutions of the elliptic operator $(-\Delta)^m$;

- Whittle–Matérn–Sobolev kernels

$$S_{m,d,\kappa}(x,y) = \frac{2^{1-m}}{(m-1)!} \kappa^{d-2m} \left(\kappa \|x-y\|_2\right)^{m-d/2} K_{m-d/2}(\kappa \|x-y\|_2)$$

  involving the Bessel function K_ν of the third kind, which are fundamental solutions of the elliptic operator $(-\Delta + \kappa^2 I)^m$ $(2m-d > 0)$.

In [B., Rossini, Schaback 2013], we introduced a new kernel ϕ for for $W_2^m(\mathbb{R}^d)$.

- we generalized both classes of these kernels by considering fundamental solutions of more general elliptic operators

$$L := \prod_{j=1}^{m}(-\Delta + \kappa_j^2 I)$$

with positive real numbers κ_j^2, $1 \leq j \leq m$ and $2m > d$.
Let

$$\kappa = \{\kappa_j^2\}_{j=1}^m \in \mathbb{R}^+ \setminus \{0\}$$

- We have provided an explicit and convenient way to compute ϕ as a *divided difference* of $S_{1,d,\kappa}$ with respect to the scale parameter vector κ.

ϕ, $m = 2$

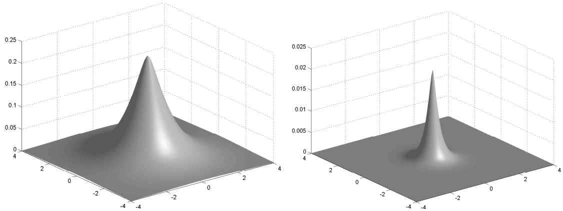


Figure: left: $\kappa_1 = 1, \kappa_2 = 2$, right: $\kappa_1 = 3, \kappa_2 = 7$,

ϕ, $m = 3$

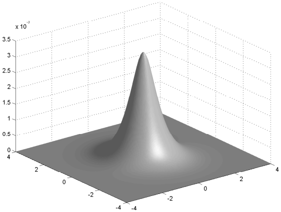


Figure: $\kappa_1 = 2, \kappa_2 = 3, \kappa_3 = 4$

Properties

- ϕ is radial strictly positive definite and decays exponentially at infinity
- $2m-d$ provide the class of regularity
- if $2m-d \geq 2$, $\phi \in C^{2m-1-d}(\mathbb{R}^d)$
- ϕ generates any basis in $W_2^m(\mathbb{R}^d)$
- in particular the lagrangian basis Λ on a set of knots $X \in \mathbb{R}^d$. Let $X = \mathbb{Z}^d$.

  Let $b = \{\phi(l)\}_{l\in\mathbb{Z}^d}$, $b \in l^1(\mathbb{Z}^d)$. Since $\hat{\phi}$ is strictly positive, by the Wiener's lemma there are unique absolutely summable coefficients $a = \{a_l\}_{l\in\mathbb{Z}^d}$ such that the cardinal function

  $$\Lambda(x) = \sum_{l\in\mathbb{Z}^d} a_l \phi(x-l) \quad \text{satisfies} \;\; \Lambda(l) = \delta_{0l}, \;\; l \in \mathbb{Z}^d$$

  and

  $$a \,|\, a * b = \delta.$$

  The vector a can be explicitly computed via an iterative algorithm (see e.g. [Bacchelli et al. 2003]) and decays exponentially.
- Λ decays exponentially at infinity.

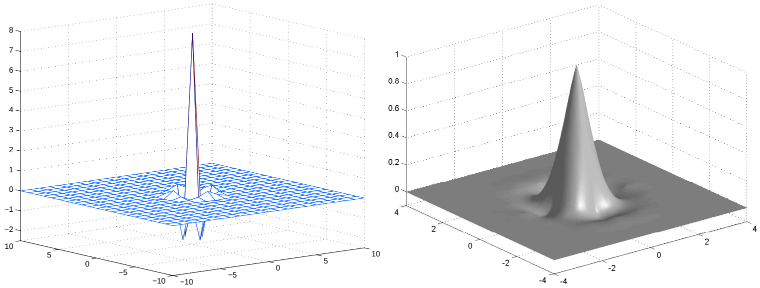


Figure: Left: a for $m = 2 : \kappa_1 = 1, \kappa_2 = 2$. Right: The Lagrangian Λ.

- The function ϕ is a scaling function [Rossini, Oslo 2012], i.e. considering the dilation matrix $A = 2I$, ϕ generates a MRA$(A, \mathbb{Z}^d)$ of $L^2(\mathbb{R}^d)$.
- We have that

$$\hat{\Lambda}(\omega) = \hat{a}(\omega)\hat{\phi}(\omega).$$

Since $a \in l^1(\mathbb{Z}^d)$, $\hat{a}(\omega) \neq 0$ in $\mathbb{T}$, according to [Madych 1992]

  - Λ is a scaling function
  - ϕ and Λ generate the same MRA

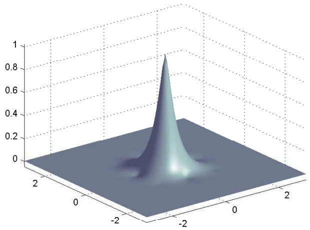


Figure: Λ with $m = 2 : \kappa_1 = 3, \kappa_2 = 7$ (left), $\kappa_1 = 10, \kappa_2 = 20$ (right)

- Λ satisfies the refinement equation

$$\Lambda(\cdot) = \sum_{l \in \mathbb{Z}^d} c_l \Lambda(2 \cdot - l),$$

with

$$c = \{\Lambda(\frac{l}{2})\}_{l \in \mathbb{Z}^d}, \quad c \in l^1(\mathbb{Z}^d).$$

- c decays exponentially.
- The sequence of the partial sums in the refinement equation converges uniformly to Λ.

Consequently, we get a convergent interpolatory subdivision scheme to a C^{2m-d-1} limit function.

Given a vector $f \in l^\infty(\mathbb{Z}^d)$, the interpolatory subdivision scheme S is defined by

$$f^0 := f \quad f^{k+1} := Sf^k, \quad k \geq 0$$

where

$$(Sf^k)_\alpha = \sum_{\beta \in \mathbb{Z}^d} c_{\alpha-2\beta} f_\beta^k.$$

Since $c \in l^1(\mathbb{Z}^d)$, the scheme converges to

$$I_f(x) = \sum_{\beta \in \mathbb{Z}^d} f_\beta \Lambda(x-\beta) \in C^{2m-d-1}(\mathbb{R}^d).$$

- The interpolant has the minimum norm in the native space
- The interpolant is the best approximation to f in the native space

- $\Lambda(x, \kappa)$ and the mask c have a numerically compact support
- $\Lambda(x, \kappa)$ depends on the values κ_j which act like tension parameters

An example

$\kappa_1 = 1,\ \kappa_2 = 2,\ e_\infty = 1e-2$

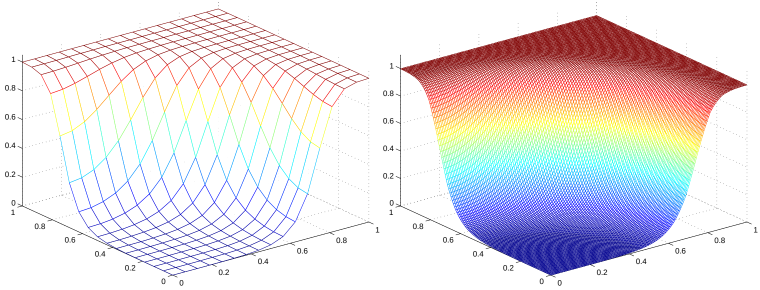


Figure: Left: f 17×17. Right: Three level of refinement

In conclusion, from $\Lambda(x, \kappa)$ we can derive a subdivision scheme that allows us to compute the surface interpolating a given data set with low computational cost.

In addition in [Bozzini, R. Canazei2012] we provided an interpolatory subdivision algorithm for non uniform meshes that

- ensures a good quality of the limit surface
- gives a flexible design capable to reproduce flat regions without undesired undulations

Example, $N = 113$, $z = (x - y)_+^6$

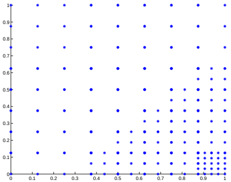


Figure: Locations of the starting vector.

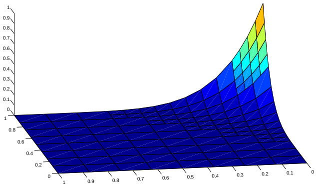
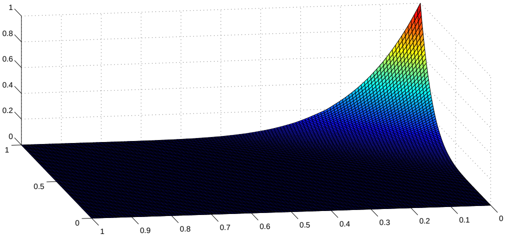


Figure: Starting vector.

$\kappa_1 = 1,\ \kappa_2 = 2,\ e_\infty = 1.9e - 003$



This new kernel can be useful also in the interpolation of functions with

- *vertical* faults
- *oblique* faults
  - capable to generate creases without undesired undulations
  - capable to reproduce cusp sections
  - capable to reproduce more general behaviours

We start from a initial vector of gridded data

$$F = \{(x_\beta, f(x_\beta), \, x_\beta \in \Omega \cap h\mathbb{Z}^2\},$$

with "large" step size h and compute the final surface via subdivision on a finer grid with step size $h_r = h/2^r$.
The basic tool is to decompose the domain Ω in the two (to fix the ideas) subdomains Ω_1 and Ω_2 given by the discontinuity curve Γ.

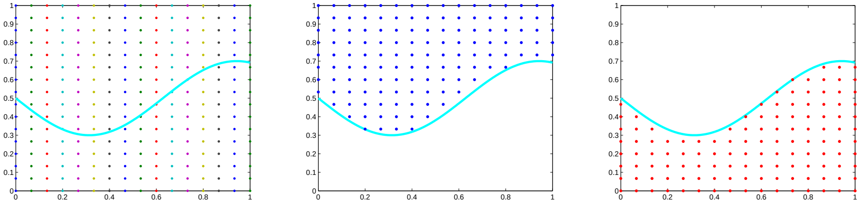


Figure: $\Omega \cap h\mathbb{Z}^2$, $\Omega_1 \cap h\mathbb{Z}^2$, $\Omega_2 \cap h\mathbb{Z}^2$

$$F \Rightarrow \begin{cases} F_1 = \{(x_\beta, f(x_\beta), x_\beta \in \Omega_1 \cap h\mathbb{Z}^2\}, \\ F_2 = \{(x_\beta, f(x_\beta), x_\beta \in \Omega_2 \cap h\mathbb{Z}^2\}. \end{cases}$$

Difficulties

- in general, the values $f(\Gamma)$ do not belong to the data set F.
- Γ is a boundary, the approximation may be poor near it;
- Having a good approximation of $f(\Gamma)$ is important for the final results and crucial in the oblique faults case

Vertical faults

In this case we can treat the two sets independently one of each other. Each set $F_l, l = 1, 2$ is extended on the whole Ω by a suitable extrapolation procedure that hopefully guarantees good values at the points of Γ and at the extended points near the boundary.

$$F_1 \Rightarrow \tilde{F}_1 = \{(x_\beta, \tilde{f}_{\beta,1}),\, x_\beta \in \Omega \cap h\mathbb{Z}^2,\, \tilde{f}_{\beta,1} = f(x_\beta),\, \beta \in \Omega_1 \cap h\mathbb{Z}^2\},$$

$$F_2 \Rightarrow \tilde{F}_2 = \{(x_\beta, \tilde{f}_{\beta,2}),\, x_\beta \in \Omega \cap h\mathbb{Z}^2,\, \tilde{f}_{\beta,2} = f(x_\beta),\, \beta \in \Omega_2 \cap h\mathbb{Z}^2\}.$$

We refine each set r times

$$\tilde{F}_1 \to \tilde{F}_1^1 \cdots \to \tilde{F}_1^r$$

$$\tilde{F}_2 \to \tilde{F}_2^1 \cdots \to \tilde{F}_2^r$$

Finally, we reassemble the discrete surfaces by cutting out the auxiliary parts

$$\tilde{F}^r = \{(x_\beta^r, \tilde{f}_{\beta,1}^r),\, x_\beta^r \in \Omega_1 \cap \frac{h}{2^r}\mathbb{Z}^2\} \cup \{(x_\beta^r, \tilde{f}_{\beta,2}^r),\, x_\beta^r \in \Omega_2 \cap \frac{h}{2^r}\mathbb{Z}^2\}$$

Examples

Example 1: $\Omega = [0,1]^2 \times [0,1]$, $N = 16 \times 16$, $h = 1/15$, $r = 3$, $\kappa = \{10, 20\}$

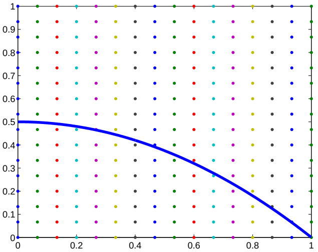


Figure: Γ and the given gridded point locations

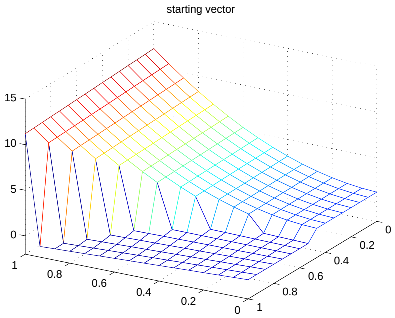
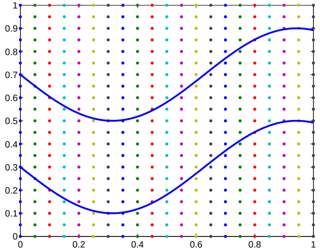


Figure: F and $\tilde{F}^k$

Maximum absolute error $e_\infty = 0.05$

- This technique can be easily extended also to the case of more than one vertical faults.

Example 2: $\Omega = [0,1]^2$, $N = 21 \times 21$, $h = 1/20$, $r = 3$, $\kappa = \{10, 20\}$



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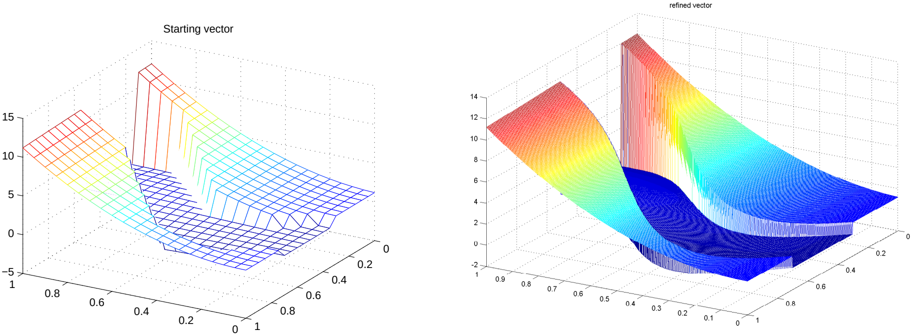
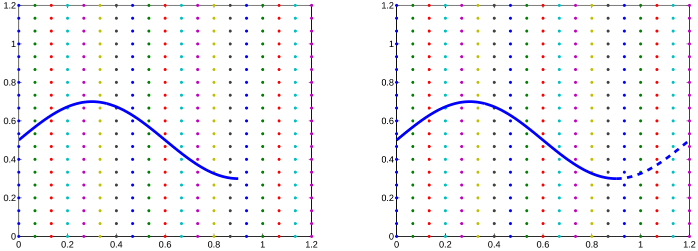


Figure: F and $\tilde{F}^k$

Maximum absolute error $e_\infty = 0.04$

- This technique can be easily extended also to the case that Γ ends (begins) at an interior point of Ω.

Example 3: $\Omega = [0, 1.2]^2$, $N = 19 \times 19$, $h = 1/15$, $r = 3$, $\kappa = \{3, 7\}$



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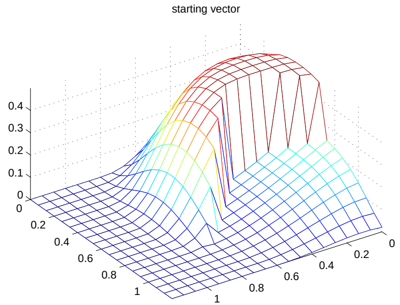


Figure: F and $\tilde{F}^k$

Maximum absolute error $e_\infty = 0.015$

Oblique faults

Few papers in the literature.

Difficulties

- The values of f at the points of Γ are generally not known but are essential to properly connect with continuity C^0 the two patches.
- we need to approximate the curve $\Gamma, f(\Gamma)$

Let

$$F_\Gamma = \{f(x_\beta, \Gamma(x_\beta)),\ \beta = 1, \dots, n\}.$$

A simple case:

- Γ coincides with a horizontal $y = y_l$ (vertical) line of the grid

$$F_\Gamma \subset F$$

Step 0: Extension of F_1 and F_2 to Ω

$$F_1 \Rightarrow \tilde{F}_1,$$
$$F_2 \Rightarrow \tilde{F}_2.$$

Step 1:
Refine 1 time the sets

$$\tilde{F}_1 \to \tilde{F}_1^1$$
$$\tilde{F}_2 \to \tilde{F}_2^1$$

and

$$F_\Gamma \to F_\Gamma^1.$$

We replace the last row of $\tilde{F}_1^1$ and the first row of $\tilde{F}_2^1$ with F_Γ^1 and repeat Step 1 r times.

Having used interpolatory schemes, when we reassemble the discrete surfaces by cutting out the auxiliary parts, the two patches are joined with continuity.

Examples

Example 4: $\Omega = [-1,1]^2$, $N = 11 \times 11$, $r = 3$ $\kappa = \{10, 20\}$, $e_\infty = 6.4e-4$

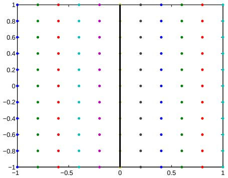
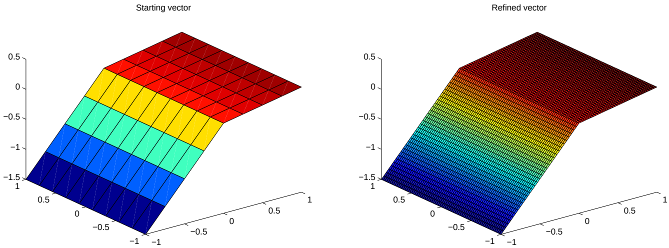


Figure: Locations of the starting vector.



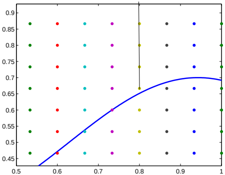
The more general case

Also in this case we split the domain in the two sub domains Ω_1 and Ω_2 and the data into the sets

$$F \Rightarrow \begin{cases} F_1 = \{(x_\beta, f(x_\beta), x_\beta \in \Omega_1 \cap h\mathbb{Z}^2\}, \\ F_2 = \{(x_\beta, f(x_\beta), x_\beta \in \Omega_2 \cap h\mathbb{Z}^2\}. \end{cases}$$

We need to approximate F_Γ

- we can choose a direction on the grid (e.g. the vertical one) and proceed line by line on the grid
- on each vertical line $x = x_l$, find the closest grid point $(x_l, y_{\bar{j}})$ to $\Gamma(x_l)$.



- Let $\theta h = \Gamma(x_l) - y_{\bar{j}}$, $0 \leq \theta < 1$, then using a Taylor expansion arrested at the first order, and approximating the partial derivative in the y direction with a backward or forward formula (e.g using three or five points), we get

$$f(x_l, \Gamma(x_l)) = f(x_l, y_{\bar{j}}) + \theta h \tilde{f}_y(x_l, y_{\bar{j}}) + O(h^2), \quad l = 1, \ldots, N$$

- we take as approximation of the values $F_\Gamma = \{f(x_l, \Gamma(x_l))\}$ the quantities

$$\tilde{F}_\Gamma = \{\tilde{f}(x_l, \Gamma(x_l)) = f(x_l, y_{\bar{j}}) + \theta h \tilde{f}_y(x_l, y_{\bar{j}}), \, l = 1, \ldots, n\}.$$

- By these values we get an approximation $\tilde{f}(x, \Gamma(x))$ of $f(x, \Gamma(x))$ by a least square technique, a Shepard's method...

Recovering the surface

- Each set $F_l, l = 1, 2$ is extended on the whole Ω by a suitable extrapolation procedure that takes in to account the values $\tilde{F}_\Gamma$.

$$F_1 \Rightarrow \tilde{F}_1, \quad F_2 \Rightarrow \tilde{F}_2$$

- We refine r times each set

$$\tilde{F}_1 \rightarrow \tilde{F}_1^1 \cdots \rightarrow \tilde{F}_1^r$$

$$\tilde{F}_2 \rightarrow \tilde{F}_2^1 \cdots \rightarrow \tilde{F}_2^r$$

- Finally, we reassemble the discrete surfaces taking care to connect the two parts with continuity but without destroying the angularities which are the peculiar features that we want to recover.

We introduce a weight function W such that

- $W \geq 0$
- its gradient is discontinuous across Γ
- its support is a small strip centered in Γ with half-width R
- W goes to zero smoothly
- e.g $W(x,y) = 1 - 3/2\left|y - \Gamma(x)\right|/R + 1/2\left|y - \Gamma(x)\right|^3/R^3$;

Then the final vector is

$$\tilde{F}^r = W(x^r,y^r)\tilde{f}(x^r,\Gamma(x^r)) + (1 - W(x^r,y^r))\begin{cases} \tilde{F}_1^r, & (x^r,y^r) \in \Omega_1 \\ \tilde{F}_2^r, & (x^r,y^r) \in \Omega_2 \end{cases}$$

Example 5:

$\Omega = [0,1]^2,\ N = 21 \times 21,\ r = 3\ \kappa = \{3,7\},\ e_\infty = 1.4e-02$

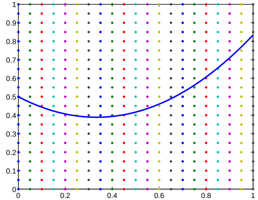


Figure: Locations of the starting vector.

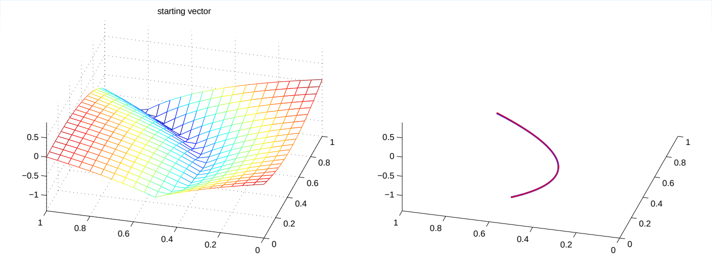
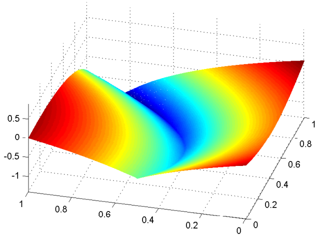


Figure: The starting vector F. $f(\Gamma)$ (blue) and its approximation (red)



THANK YOU FOR YOUR ATTENTION!

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