A Hermite Subdivision Scheme for Smooth Macro-Elements on the Powell-Sabin-12 Split

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Joint work with Tom Lyche and Nelly Villamizar
PART I:
Macro-Elements on the 12-Split
THE POWELL-SABIN SPLIT S
THE POWELL-SABIN SPLITS

6-split
THE POWELL-SABIN SPLITS

6-split

12-split
The space $S_{r,d}(\Delta_{12})$ of splines of degree $d$ and smoothness $C^r$ on the 12-split has been studied by several authors: Powell and Sabin (1977), Schumaker and Sorokina (2005), Alfeld and Schumaker, Speleers, etc.

Questions:

What is a "good" basis for $S_{r,d}$?

Can they be evaluated quickly?
The space $S^r_d(\Delta_{12})$ of splines of degree $d$ and smoothness $C^r$ on the 12-split has been studied by several authors:

- Powell and Sabin (1977)
- Schumaker and Sorokina (2005)
- Alfeld and Schumaker
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- etc.
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**Questions:**

- What is a “good” basis for $S_d^r$?
- Can they be evaluated quickly?
SPLINE SPACES ON THE 12-SPLIT
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l.b. = 39 = 39 = 39 = u.b.

\[ r = 3 \quad d = 5 \quad \text{dim} = 39 \quad \text{MDS so far: 0} \]
### Dimension Formula for the 12-Split

<table>
<thead>
<tr>
<th>$\dim S^r_d(\Delta_{12})$</th>
<th>$C^{-1}$</th>
<th>$C^0$</th>
<th>$C^1$</th>
<th>$C^2$</th>
<th>$C^3$</th>
<th>$C^4$</th>
<th>$C^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 0$</td>
<td>12</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d = 1$</td>
<td>36</td>
<td>10</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d = 2$</td>
<td>72</td>
<td>31</td>
<td>12</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d = 3$</td>
<td>120</td>
<td>64</td>
<td>30</td>
<td>16</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d = 4$</td>
<td>180</td>
<td>109</td>
<td>60</td>
<td>34</td>
<td>21</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>$d = 5$</td>
<td>252</td>
<td>166</td>
<td>102</td>
<td>61</td>
<td>39</td>
<td>27</td>
<td>21</td>
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**Theorem**

Let \(z_+ := \max\{0, z\}\). For any \(d, r \in \mathbb{Z}\) with \(d \geq 0\) and \(d \geq r \geq -1\),

\[
\dim S_d^r(\Delta_{12}) = \frac{1}{2}(r + 1)(r + 2) + \frac{9}{2}(d - r)(d - r + 1) \\
+ \frac{3}{2}(d - 2r - 1)(d - 2r)_+ + \sum_{j=1}^{d-r}(r - 2j + 1)_+
\]
Definition (macro-element)
A macro-element defined on a triangle $T$ consists of a finite-dimensional linear space $S$ of functions defined on $T$, and a set $\Lambda$ of linear functionals forming a basis for the dual of $S$.

$C^1$ quadratics
$C^3$ quintics
etc.
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- $C_3$ quintics
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$C^1$ quadratics  \hspace{1cm} $C^3$ quintics

\[\text{etc.}\]
PART II: A Hermite Subdivision Scheme
Dyn and Lyche (1998) described a Hermite subdivision scheme for evaluating $C^1$ quadratics on the 12-split.
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WHAT ABOUT THE $C^3$ QUINTICS?
$C^3$ QUINTICS: INITIALIZATION
$C^3$ QUINTICS: INITIALIZATION + SUBDIVISION
INITIALIZATION
12 \times 21 = 252 \text{ unknown } B\text{-coefficients}.
$12 \times 21 = 252$ unknown B-coefficients.
$3 \times (10 + 3) = 39$ initial conditions.
12 \times 21 = 252 \text{ unknown B-coefficients.}
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This system has a unique solution!
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270 \text{ smoothness conditions.}

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So we express the B-coefficients in terms of the initial conditions, e.g.,

\[ c_{050}^1 = c_{050}^2 = f(v_1) \]
From the beautiful book by Lai & Schumaker (2007):

\[ D_{um} \cdots D_{um} (v) = d! (d - m)! \sum_{i + j + k = d - m} c_{ijk} B_{d - m} B_{d - m} (v), \]
From the beautiful book by Lai & Schumaker (2007):

\[
D_{u_m} \cdots D_{u_1} p(v) = \frac{d!}{(d - m)!} \sum_{i+j+k=d-m} c_{ijk}^{(m)} B_{ijk}^{d-m}(v),
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INITIALIZATION: EXPLICIT FORMULAS

With $t = t_C$ and $m = m_C$ :

\[
\begin{align*}
  f_{AB} &= \frac{1}{2} f_{A+B} + \frac{7}{40} f_{A-B}^t + \frac{1}{40} f_{A+B}^{tt} + \frac{1}{640} f_{A-B}^{ttt} \\
  f_{AB}^t &= -\frac{5}{2} f_{A-B}^t - \frac{3}{4} f_{A+B}^t + \frac{3}{32} f_{A-B}^{tt} - \frac{1}{192} f_{A+B}^{ttt} \\
  f_{AB}^m &= \text{given} \\
  f_{AB}^{tt} &= -2 f_{A-B}^t - \frac{1}{2} f_{A+B}^{tt} - \frac{1}{24} f_{A-B}^{ttt} \\
  f_{AB}^{tm} &= -2 f_{A-B}^m - \frac{1}{2} f_{A+B}^{tm} - \frac{1}{24} f_{A-B}^{ttm} \\
  f_{AB}^{mm} &= f_{AAB+ABB} - \frac{1}{2} f_{A+B}^{mm} - \frac{1}{16} f_{A-B}^{ttm} \\
  f_{AB}^{ttt} &= 120 f_{A-B} + 60 f_{A+B}^t + \frac{21}{2} f_{A-B}^{tt} + \frac{3}{4} f_{A+B}^{ttt} \\
  f_{AB}^{ttm} &= -48 f_{AB}^m + 24 f_{A+B}^m + 6 f_{A-B}^{tm} + \frac{1}{2} f_{A+B}^{ttm} \\
  f_{AB}^{tmm} &= -8 f_{AAB-ABB} + 4 f_{A-B}^{mm} + \frac{1}{2} f_{A+B}^{tmm}
\end{align*}
\]
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\begin{align*}
\frac{1}{2} f_{A+B} + \frac{7}{40} f_{A-B} + \frac{1}{40} f_{A+B} + \frac{1}{640} f_{A-B} \\
- \frac{5}{2} f_{A-B} - \frac{3}{4} f_{A+B} - \frac{3}{32} f_{A-B} - \frac{1}{192} f_{A+B} \\
f_{A_B} = \text{given} \\
- \frac{2}{2} f_{A+B} - \frac{1}{2} f_{A-B} - \frac{1}{24} f_{A-B} \\
- \frac{2}{2} f_{A+B} - \frac{1}{2} f_{A-B} - \frac{1}{24} f_{A-B} \\
f_{A_B} = f_{A+B + ABB} - \frac{1}{2} f_{A+B} - \frac{1}{16} f_{A-B} \\
f_{A_B} = 120 f_{A-B} + 60 f_{A+B} + \frac{21}{2} f_{A-B} + \frac{3}{4} f_{A+B} \\
- 48 f_{A+B} + 24 f_{A+B} + 6 f_{A-B} + \frac{1}{2} f_{A+B} \\
- 8 f_{A+B - ABB} + 4 f_{A-B} + \frac{1}{2} f_{A+B} \\
f_{A_B} = \frac{5}{32} f_{C} - \frac{135}{32} f_{C} - \frac{79}{128} f_{C}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} f_{A+B} + \frac{7}{40} f_{A-B} + \frac{1}{40} f_{A+B} + \frac{1}{640} f_{A-B} \\
- \frac{5}{2} f_{A-B} - \frac{3}{4} f_{A+B} - \frac{3}{32} f_{A-B} - \frac{1}{192} f_{A+B} \\
+ 45 f_{A+B} + 36 f_{A-B} - 108 f_{A_B} \\
+ 15 f_{A+B + ABB} + 45 f_{A+B} - \frac{217}{16} f_{A+B} \\
+ \frac{25}{64} f_{A+B} + \frac{153}{16} f_{A-B} - \frac{251}{128} f_{A-B} \\
+ \frac{567}{64} f_{A+B} + \frac{43}{256} f_{A+B} + \frac{303}{512} f_{A-B} \\
+ 48(f_{BC} + f_{CA}) - 7(f_{BBC} + f_{CAA}) \\
- 90 f_{C} - 24 f_{C} - \frac{23}{8} f_{C} \\
- \frac{5}{32} f_{C} - \frac{135}{32} f_{C} - \frac{79}{128} f_{C}
\end{align*}
\]
SUBDIVISION
12 \times 21 = 252 \text{ unknown B-coefficients.}
3 \times 10 = 30 (\leq 39) \text{ initial conditions.}
288 (\geq 270) \text{ smoothness conditions.}

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So we express the B-coefficients in terms of the initial conditions, e.g.,
\[ c_{1050} = c_{2050} = f(v_1). \]
12 \times 21 = 252 \text{ unknown B-coefficients.}
3 \times 10 = 30 \ (\lt \ 39) \ \text{initial conditions.}
288 \ (\gt \ 270) \ \text{smoothness conditions.}
12 \times 21 = 252 \text{ unknown B-coefficients.}
3 \times 10 = 30 \text{ (< 39) initial conditions.}
288 \text{ (> 270) smoothness conditions.}

This system has a unique solution!
12 \times 21 = 252 \text{ unknown B-coefficients.}
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So we express the B-coefficients in terms of the initial conditions, e.g.,

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From the beautiful book by Lai & Schumaker (2007):

\[ D_{u_m} \cdots D_{u_1} p(v) = \frac{d!}{(d - m)!} \sum_{i+j+k=d-m} c_{ijk}^{(m)} B_{ijk}^{d-m}(v), \]
**Explicit Formulas: Subdivision**

With \( t = t_C \) and \( m = m_C \):

\[
\begin{align*}
    f_{AB} &= \frac{1}{2} f_{A+B} + \frac{7}{40} f_{A-B}^t + \frac{1}{40} f_{A+B}^{tt} + \frac{1}{640} f_{A-B}^{ttt} \\
    f_{AB}^t &= -\frac{5}{2} f_{A-B} + \frac{3}{4} f_{A+B}^t - \frac{3}{32} f_{A-B}^{tt} - \frac{1}{192} f_{A+B}^{ttt} \\
    f_{AB}^m &= \frac{1}{2} f_{A+B} + \frac{5}{32} f_{A-B}^m + \frac{1}{64} f_{A+B}^{tmm} \\
    f_{AB}^{tt} &= -2 f_{A-B}^t - \frac{1}{2} f_{A+B}^{tt} - \frac{1}{24} f_{A-B}^{ttt} \\
    f_{AB}^{tm} &= -2 f_{A-B}^m - \frac{1}{2} f_{A+B}^{tm} - \frac{1}{24} f_{A-B}^{ttm} \\
    f_{AB}^{mm} &= \frac{1}{2} f_{A+B}^{mm} + \frac{1}{8} f_{A-B}^{tmm} \\
    f_{AB}^{ttt} &= 120 f_{A-B} + 60 f_{A+B}^t + \frac{21}{2} f_{A-B}^{tt} + \frac{3}{4} f_{A+B}^{ttt} \\
    f_{AB}^{tmm} &= -\frac{3}{2} f_{A-B}^m - \frac{1}{4} f_{A+B}^{tmm} \\
    f_{AB}^{mmm} &= -\frac{3}{2} f_{A-B}^{mm} - \frac{1}{4} f_{A+B}^{tmm}
\end{align*}
\]

\[
\begin{align*}
    f_{AB}^{mm} &= \frac{1}{2} f_{A+B}^{mm} + \frac{1}{8} f_{A-B}^{tmm} \\
    f_{AB}^{mmm} &= -\frac{3}{2} f_{A-B}^{mm} - \frac{1}{4} f_{A+B}^{tmm}
\end{align*}
\]

\[
\begin{align*}
    f_{AB}^m &= 45 f_{A+B} + \frac{3}{16} f_{A-B}^{ttt} + \frac{15}{4} f_{A+B}^{mm} \\
    f_{AB}^{t} &= -21 f_{A+B}^m + \frac{1}{4} f_{A+B}^{mm} + 18 f_{A-B}^t \\
    f_{AB}^{tt} &= -\frac{63}{8} f_{A-B}^{tm} + \frac{5}{4} f_{A-B}^{tmm} + \frac{45}{16} f_{A+B}^{tt} \\
    f_{AB}^{tmm} &= -\frac{7}{8} f_{A+B}^{tmm} - 90 f_{C} - 48 f_{C}^m + \frac{9}{8} f_{C}^{tt} \\
    f_{AB}^{ttt} &= -\frac{21}{2} f_{C}^{mmm} - f_{C}^{mm} + \frac{1}{4} f_{C}^{tmm}
\end{align*}
\]
**EXPLICIT FORMULAS: INITIALIZATION**

With \( t = t_C \) and \( m = m_C \):

\[
\begin{align*}
\text{f}_{AB} &= \frac{1}{2} f_{A+B} + \frac{7}{40} f_{A-B} + \frac{1}{40} f_{A+B} + \frac{1}{640} f_{A-B} \\
\text{f}_t &= -\frac{5}{2} f_{A-B} - \frac{3}{4} f_{A+B} - \frac{3}{32} f_{A-B} - \frac{1}{192} f_{A+B} \\
\text{f}_m &= \text{given} \\
\text{f}_{tt} &= -2 f_{A-B} - \frac{1}{2} f_{A+B} - \frac{1}{24} f_{A-B} \\
\text{f}_{tm} &= -2 f_{A-B} - \frac{1}{2} f_{A+B} - \frac{1}{24} f_{A-B} \\
\text{f}_{mm} &= f_{AAB+ABB} - \frac{1}{2} f_{A+B} - \frac{1}{16} f_{A-B} \\
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\begin{align*}
\text{f}_{m_{mm}} &= +45 f_{A+B} + 36 f_{A-B} - 108 f_{AB} \\
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&+ \frac{25}{64} f_{A+B} + \frac{153}{16} f_{A-B} - \frac{251}{128} f_{A-B} \\
&+ \frac{567}{64} f_{A+B} + \frac{43}{256} f_{A+B} + \frac{303}{512} f_{A-B} \\
&+ 48 \left( f^{mA}_{BC} + f^{mB}_{CA} \right) + f^{mA}_{mA} \\
&+ f^{mB}_{mB} - 7 \left( f^{mA}_{mB} + f^{mB}_{mA} \right) \\
&- 90 f_C - 24 f_C - \frac{23}{8} f_{mC} \\
&- \frac{5}{32} f_{m_{mm}} - \frac{135}{32} f_{tt} - \frac{79}{128} f_{ttm}
\end{align*}
\]
PART III:
Numerical Experiments
NODAL BASIS FUNCTION: REFINEMENT LEVEL 1
NODAL BASIS FUNCTION: REFINEMENT LEVEL 2
NODAL BASIS FUNCTION: REFINEMENT LEVEL 3
NODAL BASIS FUNCTION: REFINEMENT LEVEL 4
NODAL BASIS FUNCTION: REFINEMENT LEVEL 6
SMOOTHNESS: PLOTTING DERIVATIVES
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SMOOTHNESS: PLOTTING DERIVATIVES

\( D_m \) 

\( D_t \)
SMOOTHNESS: PLOTTING DERIVATIVES

\[ D_m \rightarrow D_m \rightarrow D_m \rightarrow D_t \rightarrow D_t \rightarrow D_m \rightarrow D_m \rightarrow D_t \rightarrow D_t \]
SMOOTHNESS: PLOTTING DERIVATIVES
Flat and Phong shading: Level 1

Flat shading

Phong shading
FLAT AND PHONG SHADING: LEVEL 2

Flat shading

Phong shading
Flat and Phong Shading: Level 3

Flat shading

Phong shading
FLAT AND PHONG SHADING: LEVEL 4

Flat shading

Phong shading
Flat and Phong shading: Level 5

Flat shading

Phong shading
Flat and Phong shading: Level 6
Thank you!


Appendix
BASIS FOR DIRECTIONAL DERIVATIVES

\[ t_A + t_B + t_C = 0 \]
\[ n_A + n_B + n_C = 0 \]
\[ m_A + m_B + m_C = 0 \]

\[ m_A = \frac{1}{2} (t_B - t_C) \]
\[ t_A = \frac{2}{3} (m_C - m_B) \]
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