Analysis of Geometric Subdivision Schemes

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Erice, September 27, 2013

Joint work with Malcolm Sabin and Tobias Ewald

Standard schemes

- binary
- linear
- local
- uniform
- real-valued
- periodic grid

$$p_{2i+\sigma}^{\ell+1} = \sum_{j \le n} a_\sigma^j p_{i+j}^\ell, \quad \sigma \in \{0,1\},\ i \in \mathbb{Z},\ p_i^\ell \in \mathbb{R}.$$

Non-standard schemes

- non-stationary: Beccari, de Boor, Casciola, Dyn, Levin, Romani, Warren

$$p_{2i+\sigma}^{\ell+1} = \sum_{|j|\leq n} a_\sigma^j(\ell) p_{i+j}^\ell, \quad \sigma \in \{0,1\}^d,\ i \in \mathbb{Z}^d,\ p_i^\ell \in \mathbb{R}.$$

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$$p^{\ell+1}_{2i+\sigma} = \sum_{|j|<n} a^j_\sigma(i) p^\ell_{i+j}, \quad \sigma \in \{0,1\}^2, \; i \in \mathbb{N}^2 \times \mathbb{Z}_n, \; p^\ell_i \in \mathbb{R}^3.$$

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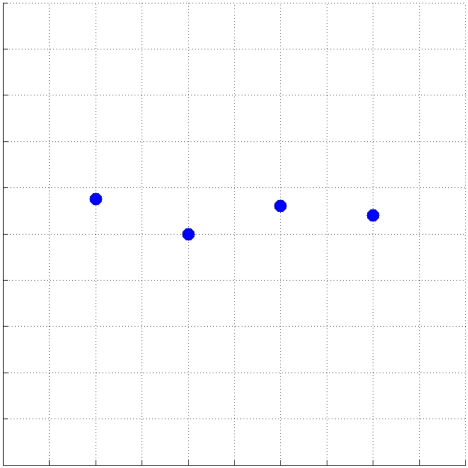
Sabin's Circle-preserving subdivision



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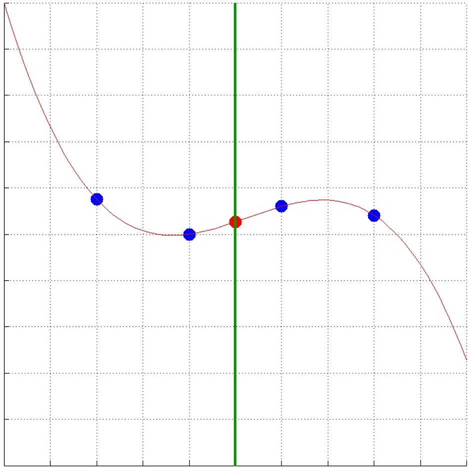

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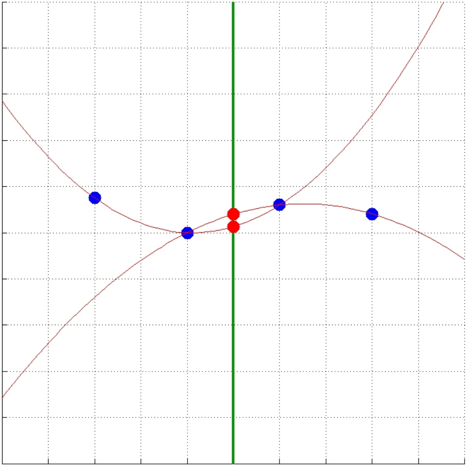

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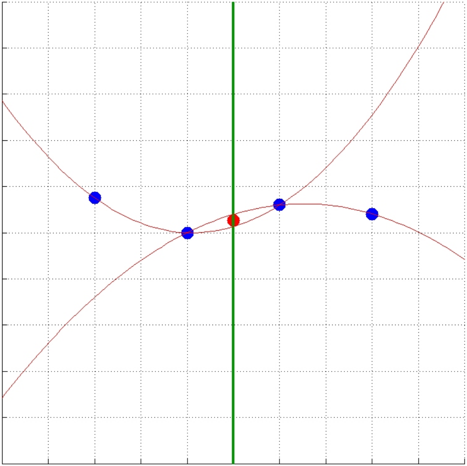

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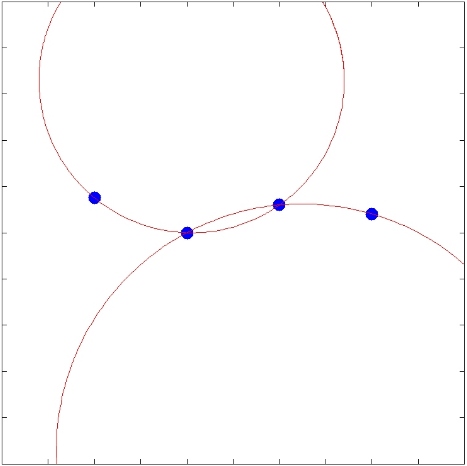

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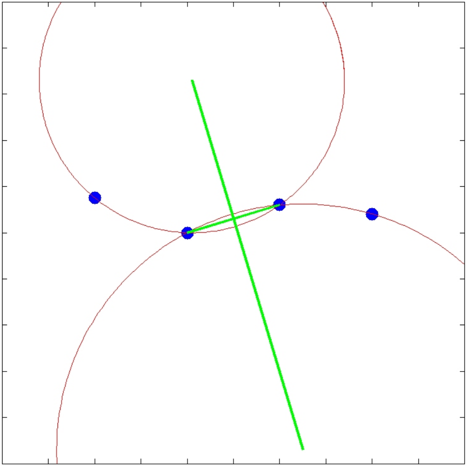

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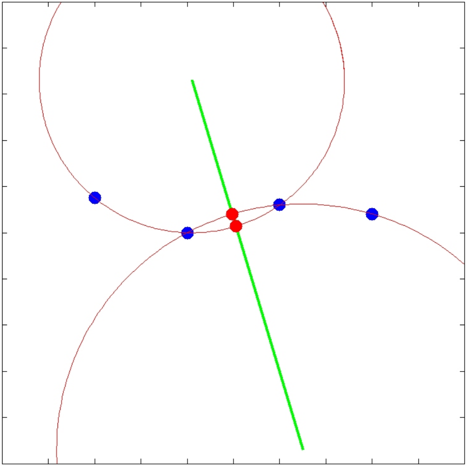

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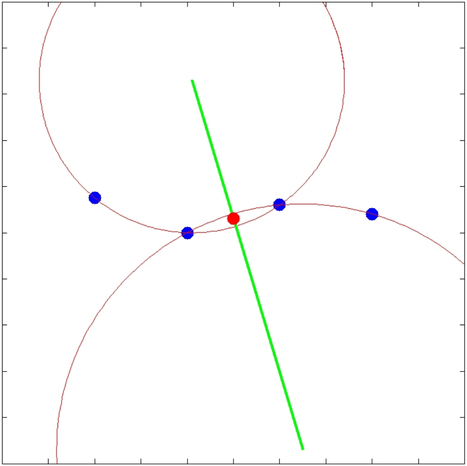

GLUE-schemes

Definition

Let $\mathbf{P} = (p_i)_{i\in\mathbb{Z}}, p_i \in \mathbb{E} := \mathbb{R}^d$. A geometric, local, uniform, equilinear subdivision scheme $\mathbf{G} : \mathbf{P}^\ell \to \mathbf{P}^{\ell+1}$ is characterized by:

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U: The same two rules (even/odd) apply everywhere,

$$p^{\ell+1}_{2i+\sigma} = g_\sigma(p^\ell_i, \ldots, p^\ell_{i+m}), \quad \sigma \in \{0, 1\}.$$

The functions g_σ are $C^{1,1}$ in a neighborhood of linear data.

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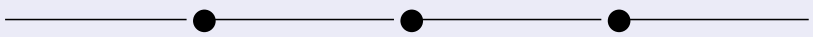
$$\mathbf{G}(\mathbf{E}) = \mathbf{E}/2$$



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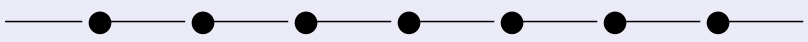

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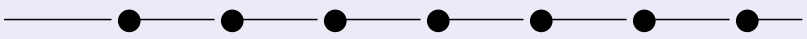

Basics: matrix-like formalism

- Analogous to the representation of linear schemes in terms of pairs of matrices, there exist functions $\mathbf{g}_\sigma$ such that

$$\mathbf{p}^{\ell+1}_{2i+\sigma} = \mathbf{g}_\sigma(\mathbf{p}^\ell_i),$$

  where $\mathbf{p}^\ell_i = [p^\ell_i; \dots ; p^\ell_{i+n-1}]$ are subchains of $\mathbf{P}^\ell$ of length n.

- Constant chains are fixed points,

$$\mathbf{g}_\sigma(\mathbf{p}) = \mathbf{p} \quad \text{if} \quad \Delta\mathbf{p} = 0.$$

- Composition of functions $\mathbf{g}_\sigma$ is denoted by

$$\mathbf{g}_\Sigma = \mathbf{g}_{\sigma_\ell} \circ \cdots \circ \mathbf{g}_{\sigma_1}, \quad \Sigma = [\sigma_1, \dots, \sigma_\ell],\ |\Sigma| = \ell.$$

- Let $\mathbf{e} := [e; \dots ; ne]$. Then

$$\mathbf{g}_\Sigma(\mathbf{e}) = 2^{-|\Sigma|}\mathbf{e} + \tau_\Sigma e.$$

Basics: spaces of chains

- $\mathbb{E}^{\mathbb{Z}} := \mathbb{R}^{d\times\mathbb{Z}}$ is the space of infinite chains in $\mathbb{R}^d$.
- $\mathbb{E}^n := \mathbb{R}^{d\times n}$ is the space of chains with n vertices in $\mathbb{R}^d$.
- $\mathbb{L}^n := \{\mathbf{p} \in \mathbb{E}^n : \Delta^2\mathbf{p} = 0\}$ is the space of linear chains.
- $\Pi : \mathbb{E}^n \to \mathbb{L}^n$ is the orthogonal projector onto $\mathbb{L}^n$.
- For $\mathbf{P} \in \mathbb{E}^{\mathbb{Z}}$, let

$$\|\mathbf{P}\| := \sup_i \|p_i\|_2, \quad |\mathbf{P}|_1 := \|\Delta\mathbf{P}\|, \quad |\mathbf{P}|_2 := \|\Delta^2\mathbf{P}\|.$$

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Basics: relative distortion

- The relative distortion of some chain $\mathbf{p} \in \mathbb{E}^n$ is defined by

$$\kappa(\mathbf{p}) := \begin{cases} \dfrac{|\mathbf{p}|_2}{|\Pi \mathbf{p}|_1} & \text{if } |\Pi \mathbf{p}|_1 \neq 0 \\ \infty & \text{if } |\Pi \mathbf{p}|_1 = 0. \end{cases}$$

- Invariance under similarities,

$$\kappa(\mathbf{p}) = \kappa(S(\mathbf{p})), \quad S \in \mathcal{S}(\mathbb{E}).$$

- Distortion of infinite chain,

$$\kappa(\mathbf{P}) := \sup_{i \in \mathbb{Z}} \kappa(\mathbf{p}_i), \quad \mathbf{p}_i = [p_i; \ldots; p_{i+n-1}].$$

- Distortion sequence generated by subdivision,

$$\kappa_\ell := \kappa(\mathbf{P}^\ell), \quad \mathbf{P}^\ell := \mathbf{G}^\ell(\mathbf{P}).$$

Straightening

Definition

The chain $\mathbf{P}$ is

- **straightened** by $\mathbf{G}$ if κ_ℓ is a null sequence;
- **strongly straightened** by $\mathbf{G}$ if κ_ℓ is summable;
- **straightened** by $\mathbf{G}$ **at rate** α if $2^{\ell\alpha}\kappa_\ell$ is bounded.

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Lemma

Let $\mathbf{G}$ be a GLUE-scheme. If the chain $\mathbf{P}$ is

- *straightened by $\mathbf{G}$, then $|\mathbf{P}|_1 \le Cq^\ell$ for any $q > 1/2$;*
- *strongly straightened by $\mathbf{G}$, then $|\mathbf{P}|_1 \le Cq^\ell$ for any $q = 1/2$;*
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Proof:

- induction on $|\Sigma|$
- q-Pochhammer symbol

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Theorem (R. 2013)

Let $\mathbf{G}$ *be a GLUE-scheme. If the chain* $\mathbf{P}$ *is*

- *straightened by* $\mathbf{G}$*, then* $\mathbf{P}^\ell$ **converges** *to a continuous limit curve;*
- *strongly straightened by* $\mathbf{G}$*, then the limit curve is* C^1 *and regular;*
- *straightened by* $\mathbf{G}$ *at rate* α*, then the limit curve is* $C^{1,\alpha}$ *and regular.*

Convergence

- Let φ be a C^k-function which
  - has compact support;
  - constitutes a partition of unity, $\sum_j \varphi(\cdot - j) = 1$.
- Associate a curve Φ^ℓ to the chain $\mathbf{P}^\ell$ at stage ℓ by

$$\Phi^\ell[\mathbf{P}^\ell] := \sum_{\ell \in \mathbb{Z}} p_j^\ell \varphi(2^\ell \cdot - j)$$

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  is well defined, continuous, and independent of φ.
- If $\Phi^\ell[\mathbf{P}^\ell]$ is Cauchy in C^k, then the limit curve $\Phi[\mathbf{P}]$ is C^k.
- Use modulus of continuity to establish Hölder exponent.

Proximity

- Given a GLUE-schem $\mathbf{G}$, choose a linear subdivision scheme $\mathbf{A}$ with **equal shift**, i.e., $\mathbf{G}(\mathbf{E}) = \mathbf{A}\mathbf{E} = (\mathbf{E} + \tau e)/2$.
- Schemes $\mathbf{G}$ and $\mathbf{A}$ differ by **remainder $\mathbf{R}$**,

$$\mathbf{R}(\mathbf{P}) := \mathbf{G}(\mathbf{P}) - \mathbf{A}\mathbf{P}.$$

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$$\mathbf{R}(\mathbf{P}) := \mathbf{G}(\mathbf{P}) - \mathbf{AP}.$$

- Choose φ as limit function of $\mathbf{A}$ correspondig to **Dirac** data $\delta_{j,0}$ to define curves $\Phi^\ell[\mathbf{P}^\ell]$ at level ℓ.
- Curves at levels ℓ and $\ell + r$ differ by

$$\left|\partial^j \left(\Phi^{\ell+r}[\mathbf{P}^{\ell+r}] - \Phi^\ell[\mathbf{P}^\ell]\right)\right|_\infty \leq c \sum_{i=\ell}^{\infty} 2^{ij} |\mathbf{R}(\mathbf{P}^i)|_0.$$

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- Use bound

$$|\mathbf{R}(\mathbf{P}^i)|_0 \leq c\kappa_i q^i$$

with $q = 1/2$ in case of strong straightening, and $q = 2/3$ otherwise.

Checks for straightening

For applications, we need explicit values α, δ such that **P** is straightened by **G** at rate α whenever $\kappa(\mathbf{P}) \leq \delta$.

Checks for straightening

Lemma

Let

$$\Gamma_\ell[\delta] := \sup_{0<|\mathbf{d}|_2\le\delta} \frac{\kappa_\ell(\mathbf{e}+\mathbf{d})}{|\mathbf{d}|_2}.$$

If $\Gamma_\ell[\delta] < 1$ *for some* $\ell \in N$*, then* **P** *is straightened by* **G** *at rate*

$$\alpha = -\frac{\log_2 \Gamma_\ell[\delta]}{\ell}$$

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+ A rigorous upper bound on $\Gamma_\ell[\delta]$ can be established using mean value theorem and interval arithmetics.
– The larger δ, the poorer α.

Checks for straightening

Theorem (R. 2012)

Let

$$\Gamma_\ell[\delta] := \sup_{0<|\mathbf{d}|_2\le\delta} \frac{\kappa_\ell(\mathbf{e}+\mathbf{d})}{|\mathbf{d}|_2} \quad \textit{and} \quad \Gamma_k[\delta,\gamma] := \max_{\delta\le|\mathbf{d}|_2\le\gamma} \frac{\kappa_k(\mathbf{e}+\mathbf{d})}{|\mathbf{d}|_2}.$$

If $\Gamma_\ell[\delta] < 1$ *for some* $\ell \in N$, *and* $\Gamma_k[\delta,\gamma] < 1$ *for some* $k \in \mathbb{N}$, *then* $\mathbf{P}$ *is straightened by* $\mathbf{G}$ *at rate*

$$\alpha = -\frac{\log_2 \Gamma_\ell[\delta]}{\ell}$$

whenever $\kappa(\mathbf{P}) \le \gamma$.

Checks for straightening

Theorem (R. 2012)

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whenever $\kappa(\mathbf{P}) \le \gamma$*.*

- Rigorous upper bounds on $\Gamma_\ell[\delta]$ and $\Gamma_k[\delta,\gamma]$ via interval arithmetics.
- Choose δ as small as possible to get good α.
- Choose γ as large as possible to get good range of applicability.

- In general, the derivative of a function $\mathbf{g} : \mathbb{E}^n \to \mathbb{E}^n$ is represented by $n \times n$ matrices of dimension $d \times d$, each.

Differentiation

- In general, the derivative of a function $\mathbf{g} : \mathbb{E}^n \to \mathbb{E}^n$ is represented by $n \times n$ matrices of dimension $d \times d$, each.
- By property G, the derivative of $\mathbf{g}_\sigma$ at $\mathbf{e}$ has the special form

$$D\mathbf{g}_\sigma(\mathbf{e}) \cdot \mathbf{q} = A_\sigma \mathbf{q}\, \Pi^n + B_\sigma\, \mathbf{q}\, \Pi^t, \quad \sigma \in \{0, 1\},$$

where A_σ, B_σ are $(n \times n)$-matrices, and

$$\Pi^t := \text{diag}[1, 0, \ldots, 0], \quad \Pi^n := \text{diag}[0, 1, \ldots, 1]$$

are $(d \times d)$-matrices representing orthogonal projection onto the x-axis and its orthogonal complement.

- Let $\mathbf{A} = (A_0, A_1)$ and $\mathbf{B} = (B_0, B_1)$ denote the linear subdivision schemes corresponding to normal and tangential direction.

Inheritance of $C^{1,\alpha}$-regularity

Theorem (R. 2013)

Let the linear schemes **A** *and* **B** *be* $C^{1,\alpha}$ *and* $C^{1,\beta}$*, resp. If* **P** *is straightened by* **G***, then the limit curve* $\Phi[\mathbf{P}]$ *is* $C^{1,\min(\alpha,\beta)}$.

Proof: Show that

$$\lim_{\delta\to 0}\liminf_{\ell\to\infty}\big(\Gamma_\ell[\delta]\big)^{1/\ell} \le \max\big(\mathrm{jsr}(A_0^2, A_1^2), \mathrm{jsr}(B_0^2, B_1^2)\big).$$

Locally linear schemes

Definition

A GLUE-scheme $\mathbf{G}$ is called **locally linear** if there exist $(n \times n)$-matrices A_0, A_1 such that

$$D\mathbf{g}_\sigma(\mathbf{e}) \cdot \mathbf{q} = A_\sigma \mathbf{q}\, \Pi^n + B_\sigma\, \mathbf{q}\, \Pi^t.$$

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In this case the linear scheme $\mathbf{A} = (A_0, A_1)$ is called the **linear companion** of $\mathbf{G}$.

- For $d = 1$, any GLUE-scheme $\mathbf{G}$ is locally linear.
- For $d \geq 2$, the scheme $\mathbf{G}$ is locally linear if $\mathbf{A} = \mathbf{B}$.
- Circle-preserving subvdivion is locally linear, and the standard four-point scheme is its linear companion.

Inheritance of $C^{2,\alpha}$-regularity

Theorem (R. 2013)

Let $\mathbf{G}$ *be locally linear, and let the linear companion* $\mathbf{A}$ *be* $C^{2,\alpha}$*. If* $\mathbf{P}$ *is straightened by* $\mathbf{G}$*, then the limit curve* $\Phi[\mathbf{P}]$ *is* $C^{2,\alpha}$*.*

Proof:

- Use basic limit function φ of $\mathbf{A}$ to define curves $\Phi^\ell[\mathbf{P}^\ell]$.
- Use bound

$$|\mathbf{R}(\mathbf{P})|_0 \leq c\kappa(\mathbf{P})|\mathbf{P}|_2$$

on the remainder $R(\mathbf{P}) := \mathbf{G}(\mathbf{P}) - \mathbf{AP}$.

Inheritance of $C^{3,\alpha}$-regularity

Inheritance of $C^{3,\alpha}$-regularity

...cannot be expected!

A counter-example

Consider

$$g_0(p_i^\ell,\dots,p_{i+3}^\ell) = \frac{6}{32}\,p_i^\ell + \frac{20}{32}\,p_{i+1}^\ell + \frac{6}{32}\,p_{i+2}^\ell + \frac{\|\Delta^2 p_i^\ell\|}{\|\Delta p_i^\ell\|}\,\Delta^2 p_i^\ell$$

$$g_1(p_i^\ell,\dots,p_{i+3}^\ell) = \frac{1}{32}\,p_i^\ell + \frac{15}{32}\,p_{i+1}^\ell + \frac{15}{32}\,p_{i+2}^\ell + \frac{1}{32}\,p_{i+3}^\ell$$

The scheme is locally linear with A_0, A_1 representing quintic B-spline subdivision. However, limit curves $\Phi^\infty[\mathbf{P}]$ are not C^4, and not even C^3.

Conclusion

- Geometric subdivision schemes deserve attention.
- Results apply to a wide range of algorithms.
- Hölder continuity of first order can be established rigorously by means of a universal computer program (at least in principle, runtime may be a problem).
- For locally linear schemes, Hölder-regularity of second order can be derived from a linear scheme, defined by the Jacobians at linear data.
- Regularity of higher order requires new concepts.