Intrinsic Supersmoothness

Tatyana Sorokina

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plan of the talk

1. What is supersmoothness?

2. History and contributions

3. What is the supersmoothness good for?

4. What about non-polynomial “splines”? Can they have supersmoothness?

5. Some more detailed results

6. Conclusions and conjectures

what is supersmoothness

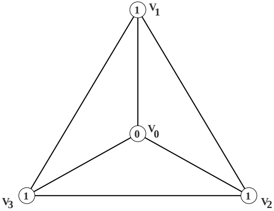
A C^r-differentiable piecewise polynomial function on a n-dimensional simplicial complex $\Delta \subseteq \mathbb{R}^n$ is called a *spline*. Let $S_d^r(\Delta)$ denote the vector space of C^r splines on a fixed Δ.

what is supersmoothness

A C^r-differentiable piecewise polynomial function on a n-dimensional simplicial complex $\Delta \subseteq \mathbb{R}^n$ is called a *spline*. Let $S_d^r(\Delta)$ denote the vector space of C^r splines on a fixed Δ.

Let $\sigma \in \Delta$ be a k-dimensional simplex in Δ, $k < n$. If for any $s \in S_d^r(\Delta)$, it follows that $s \in C^\mu(\sigma)$, where $\mu > r$, then we say that $S_d^r(\Delta)$ has supersmoothness μ at σ.

what is supersmoothness: Clough-Tocher example



$$s \in S^1_d(\Delta) \quad \rightarrow \quad s \in C^2(v_0)$$

$$dim\ S^1_2(\Delta) = dim\ P_2 = 6$$

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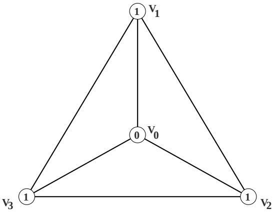


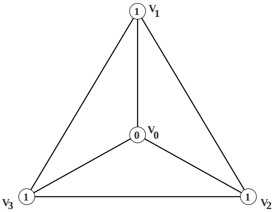
$s \in S_d^1(\Delta) \quad \rightarrow \quad s \in C^2(v_0)$

$dim\ S_2^1(\Delta) = dim\ P_2 = 6$

$s \in S_d^3(\Delta) \quad \rightarrow \quad s \in C^5(v_0)$

$dim\ S_5^3(\Delta) = dim\ P_5 = 21$

what is supersmoothness: Clough-Tocher example



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$s \in S^3_d(\Delta) \quad \rightarrow \quad s \in C^5(v_0)$

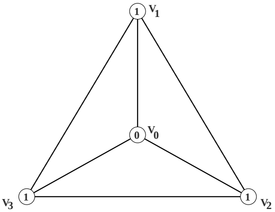
$dim\ S^3_5(\Delta) = dim\ P_5 = 21$

BUT there is a big difference between these two examples

$C^2(v_0)$ is true C^2 differentiability at v_0, while $C^5(v_0)$, for $d > 5$, is equality of all partial derivatives of order five at v_0.

Why?

what is supersmoothness: Clough-Tocher example



$$s \in S_d^1(\Delta) \quad \to \quad s \in C^2(v_0)$$

$$dim\ S_2^1(\Delta) = dim\ P_2 = 6$$

$$s \in S_d^3(\Delta) \quad \to \quad s \in C^5(v_0)$$

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$C^2(v_0)$ is true C^2 differentiability at v_0, while $C^5(v_0)$, for $d > 5$, is equality of all partial derivatives of order five at v_0.

Why?

Because if s were order **five** differentiable at v_0 then it would have been order **four** differentiable in a neighborhood of v_0.

what is supersmoothness: example

$$s(x,y) = \begin{cases} 0 & \text{if} \quad x \geq 0, \\ y^2 & \text{if} \quad x < 0, \end{cases}$$

Such $s(x,y)$ is not even continuous on $\mathbf{R}^2$. However, $s \in C^0\big((0,0)\big)$ and, moreover,

$$\frac{\partial s}{\partial x}(0,0) = \frac{\partial s}{\partial y}(0,0) = 0.$$

Thus, s has supersmoothness one at the origin but not differentiability of order one at the origin.

Continuity of this $C^{-1}(\mathbf{R}^2)$ spline at the origin is of course the true continuity.

what is supersmoothness

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- μ does NOT depend on d, it depends on Δ and r
- univariate splines have no supersmoothness
- supersmoothness is not always "superdifferentiability"

history and contributions

1980. The Clough-Tocher example is discovered.

G. Farin, Bézier polynomials over triangles; Report TR/91, Dept. of Mathematics, Brunel University, Uxbridge, UK, 1980

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P. Alfeld, A trivariate Clough-Tocher scheme for tetrahedral data, *Computer Aided Geometric Design* **1**, 1984 169–181

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2013. Do all multivariate splines have some supersmoothness?

T. Sorokina, Supersmoothness of bivariate splines and geometry of the underlying partition, submitted, 2013, see my webpage.

2013. Do other functions have supersmoothness?

B. Shekhtman and T. Sorokina, Intrinsic Supersmoothness, submitted, 2013, arXiv:1302.5102.

computing dimensions: dim $S_2^1(\Delta_n) = ?$

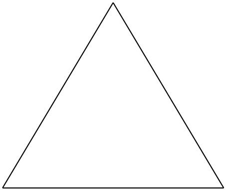

Figure : $dim\ S_2^1(\Delta_1) = 6$

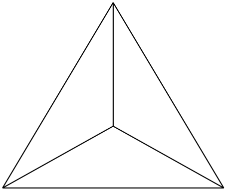

Figure : $dim\ S_2^1(\Delta_3) = 6$

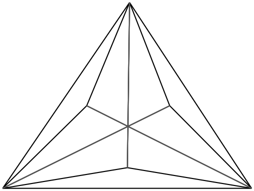

Figure : $dim\ S_2^1(\Delta_9) = 6$

bivariate splines: more toy examples

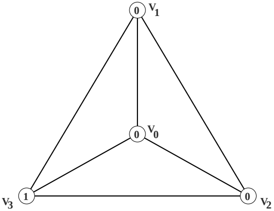

Figure : $dim\ S_1^{\mathcal{T}} = 3$

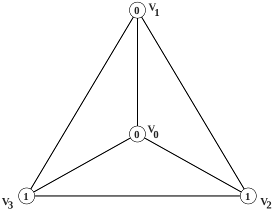

Figure : $dim\ S_1^{\mathcal{T}'} = 3$

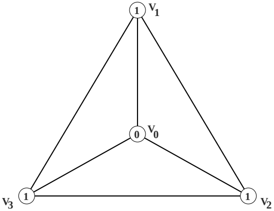

Figure : $dim\ S_1^{\mathcal{T}''} = 3$

can we do better than algebraic geometers?

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Smooth planar r-splines of degree $2r$

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Abstract

Alfeld and Schumaker [Numer. Math. 57 (1990) 651–661] give a for mula for the dimension of the space of piecewise polynomial functions (splines) of degree d and smoothness r on a generic triangulation of a planar simplicial complex Δ (for $d \geqslant 3r+1$) and any triangulation (for $d \geqslant 3r+2$). In Schenck and Stiller [Manuscripta Math. 107 (2002) 43–58], it was conjectured that the Alfeld–Schumaker formula actually holds for all $d \geqslant 2r + 1$. In this note, we show that this is the best result possible; in particular, there exists a simplicial complex Δ such that for any r, the dimension of the spline space in degree $d = 2r$ is not given by the formula of Alfeld and Schumaker [Numer. Math. 57 (1990) 651–661]. The proof relies on the explicit computation of the nonvanishing of the first local cohomology module described in Schenck and Stillman [J. Pure Appl. Algebra 117 & 118 (1997) 535–548].
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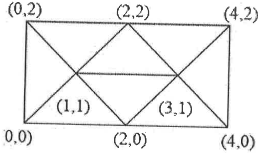
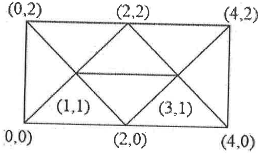
MSC: primary 13D40; secondary 52B20

Keywords: Simplicial complex; Bivariate spline; Hilbert function



$S^r_d(\Delta)$ for $r \leq 2d$
$dim\ S^r_d(\Delta) = ?$



$S_d^r(\Delta)$ for $r \leq 2d$
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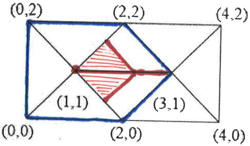
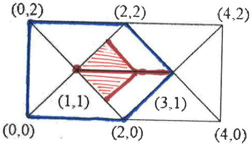
Then supersmoothness implies ....



$s \in S_d^r(\textit{left blue pentagon})$ implies s has supersmoothness $\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor$ at $(1,1)$ across the red edge ONLY



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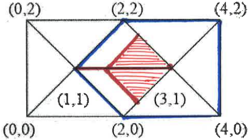

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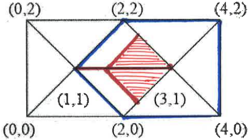
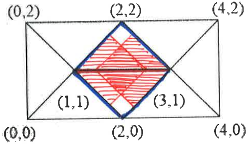
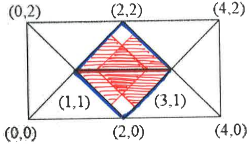


$s \in S_d^r(\textit{right blue pentagon})$ implies s has supersmoothness $\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor$ at $(3,1)$ across the red edge ONLY

Then the overlap implies ....



$s \in S_d^r(\textit{blue rhombus})$ implies s has supersmoothness $\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor$ across the red edge. Thus $S_d^r(\Delta) = S_d^{r,\mu}(\Delta)$



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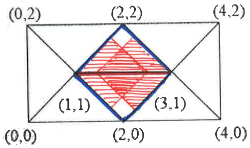
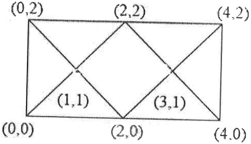
Then we play this game of again and



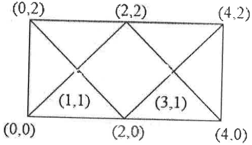
$s \in S_d^r(\mathit{blue\ rhombus})$ implies s has supersmoothness $\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor$ across the red edge. Thus $S_d^r(\Delta) = S_d^{r,\mu}(\Delta)$

Then we play this game of again and

μ becomes $r + \left\lfloor \frac{r+3}{2} \right\rfloor$. We play this game again and again and ....



the true partition emerges.....
there has never been a red edge



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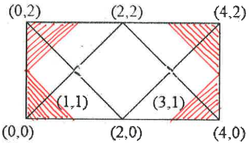
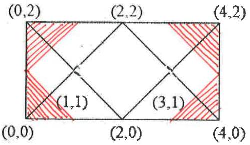
Then we apply the usual Bernstein-Bézier techniques and ....



• red smoothness conditions in the corners can be considered independently of those in the white area

• since $d \leq 2r$, the smoothness conditions inside the white area are so tight that it is just one polynomial.



- red smoothness conditions in the corners can be considered independently of those in the white area
- since $d \leq 2r$, the smoothness conditions inside the white area are so tight that it is just one polynomial.

Then we simply count the domain points (too boring to present it here) and get the exact dimension.

sometimes one has to use algebraic geometry

Theorem

For all integers $d \geq 0$ and $n \geq 1$,

$$dim\ S_d^1(A_n) = \binom{d+n}{n} + n\binom{d-1}{n},$$

where A_n is the Alfeld split of a simplex in $\mathbf{R}^n$ with one interior split point into $n+1$ subsimplices.

A. Kolesnikov and T. Sorokina, Multivariate C^1-continuous splines on the Alfeld split of a simplex, submitted, 2013, see my webpage.

The proof would have been impossible without

Theorem

Let $s \in S_d^1(A_n)$. Then $s \in C^n(v_0)$.

what about non-polynomial splines

B. Shekhtman, T. Sorokina, Intrinsic supersmoothness, 2013, submitted, arXiv:1302.5102

Using only standard tools from multivariate calculus, we show that if we continuously glue two smooth functions along a curve with a "corner", the resulting continuous function must be differentiable at the corner, as if to compensate for the singularity of the curve. Moreover, locally, this property characterizes non-smooth curves. We also generalize this phenomenon to higher order derivatives. In particular, this shows that supersmoothness has little to do with properties of polynomials.

T. Sorokina, Supersmoothness of bivariate splines and geometry of the underlying partition, 2013, submitted

Using only standard Bernstein-Bézier tools, we show that many types of supersmoothness have everything to do with polynomial nature of splines.

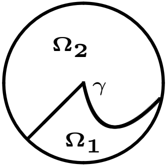
supersmoothness at singular point

Theorem (2012)

Let $\gamma \subset \mathbb{R}^2$ be the trace of a Jordan arc that divides the open disk Ω into two subsets Ω_1 and Ω_2. Let γ is not smooth at $P \in \gamma$. Let f_1, f_2 be C^1 functions on Ω continuously glued along γ, that is, let

$$F(x,y) := \begin{cases} f_1(x,y) & \text{if} \quad (x,y) \in \Omega_1, \\ f_2(x,y) & \text{if} \quad (x,y) \in \Omega_2, \end{cases}$$

be a continuous function on Ω. Then the piecewise function F is differentiable at P, that is, $\nabla f_1(P) = \nabla f_2(P)$.



local characterization of non-smooth curves

Theorem (2012)

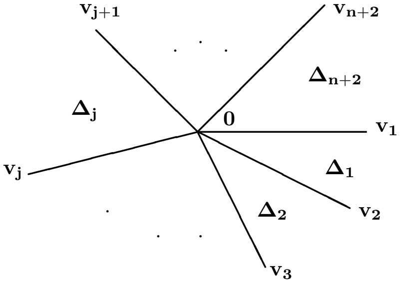
The trace of a Jordan arc γ is smooth at P if and only if there exists a neighborhood U of P and a function h continuously differentiable on U such that

$$h(x, y) = 0 \ \text{ if } \ (x, y) \in \gamma \cap U, \quad \text{and} \ \ \nabla h(P) \neq \mathbf{0}.$$

supersmoothness of higher derivatives

Theorem (2012)

Let functions $f_1, \ldots, f_{n+2}$, be n times continuously differentiable on Ω and let F be defined piecewise on each sector Δ_j by $F\,|_{\Delta_j} := f_j$, $j = 1, \ldots, n+2$. If $F \in C^n(\Omega)$ then F has all derivatives of order $n+1$ at the origin, that is, $F \in C^{n+1}(\mathbf{0})$, $n \geq 0$.



the book on splines

M. J. Lai, L. L. Schumaker, *Spline Functions on Triangulations*, Cambridge University Press (Cambridge), 2007.

bivariate splines: dim on a cell

Let a cell $\triangle$ have n edges, $\{e_i\}_{i=1}^n$, whose slopes are $\{a_i\}_{i=1}^n$, respectively. We note that any cell can be rotated so that the slopes are defined. Given a set $\mathcal{T}$ of strongly supported smoothness functionals associated with $\triangle$

$$\dim S_d^{\mathcal{T}}(\triangle) = \sum_{i=1}^{n}\sum_{j=0}^{d}(j - r_{i,j}) + \sum_{j=0}^{d}(j + 1 - \varepsilon_j)_+,$$

where

$$\varepsilon_j := \sum_{i=1}^{n} m_{i,j},$$

$$m_{i,j} := \begin{cases} 0, & \text{if there exists } l \text{ with } a_i = a_l \text{ and } r_{l,j} < r_{i,j}, \\ 0, & \text{if there exists } l > i \text{ with } a_i = a_l \text{ and } r_{l,j} = r_{i,j}, \\ j - r_{i,j}, & \text{otherwise.} \end{cases}$$

Theorem (2013)

Let $S_d^{\mathcal{T}}(\triangle)$ with strongly supported $\mathcal{T}$ be defined on a cell $\triangle$ with n edges. Given $\mu \in \{1, \dots, n\}$ and $\nu \in \{0, \dots, d\}$, let $r_{\mu,\nu} < \nu$ be the smoothness value in $\mathcal{T}$ associated with the edge e_μ on level ν. If $\mathcal{T}' := \mathcal{T} \cup \tau_{\nu,e_\mu}^{r_{\nu,\mu}+1}$ remains strongly supported, then $S_d^{\mathcal{T}}(\triangle) = S_d^{\mathcal{T}'}(\triangle)$ if and only if

$$\varepsilon_\nu \leq \nu + 1,$$

and either

(i) e_μ has no collinear counterpart or

(ii) e_μ has a collinear counterpart with strictly higher smoothness value on level ν.

bivariate splines: more examples

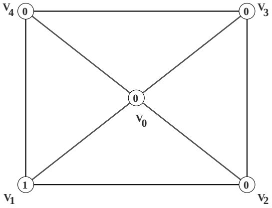


Figure : $dim\ S_1^{\mathcal{T}} = 4$

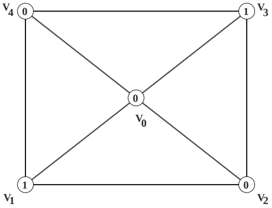


Figure : $dim\ S_1^{\mathcal{T}'} = 4$

bivariate splines: more examples

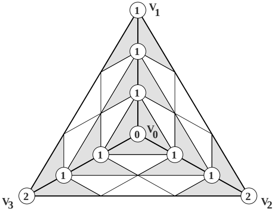


Figure : $dim\ S_3^{\mathcal{T}} = 10$

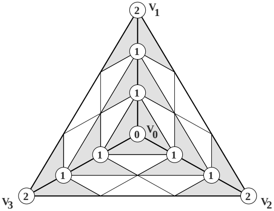
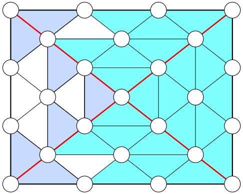
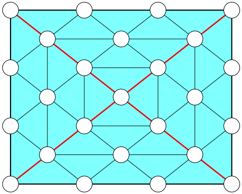


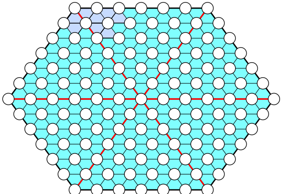
Figure : $dim\ S_3^{\mathcal{T}'} = 10$

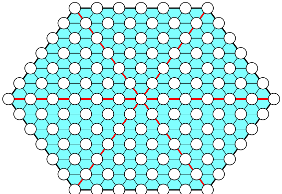
Example: $\mathbf{r} = \{(1,2),(1,2)\}$, $\bar{\mathbf{r}} = \{2,2\}$, $d = 3$, $dim = 12$





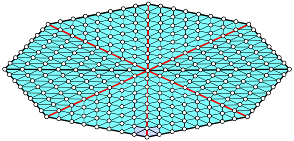
Example: $\mathbf{r} = \{(4,5),(4,5),(3,4)\}$, $\bar{\mathbf{r}} = \{5,5,4\}$, $d = 6$, $dim = 33$

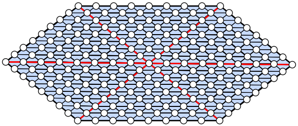




example dim=48

Two non-collinear edges have smoothness 7 and 6. Three pairs of collinear edges have pairs of smoothness $(7,7),(5,7)$, $(6,7)$. Then for $d=8$ the two non-collinear edges can be removed.





In fact, the new space $S_8^{\mathbf{r}}$ with $\mathbf{r}=\{(7,7),(5,7),(6,7)\}$ is the same as S_8^7.

Theorem (2013)

Let $\triangle$ be a cell with m slopes and m pairs of collinear edges. Suppose $\mathcal{T}$ is defined by the following smoothness conditions: for each pair of collinear edges $(e_i, \tilde{e}_i)$, let (r_i, ρ_i) be the smoothness across e_i and $\tilde{e}_i$, respectively, with the convention $r_i \leq \rho_i \leq d$. Suppose $\mathcal{T}'$ is defined by the following smoothness conditions: for each pair of collinear edges $(e_i, \tilde{e}_i)$, let ρ_i be the smoothness across both of them. Then

$$S_d^{\mathcal{T}}(\triangle) = S_d^{\mathcal{T}'}(\triangle), \quad \textit{whenever} \quad d \leq d^* := \left\lfloor \frac{\sum_{i=1}^m r_i + 1}{m-1} \right\rfloor .$$

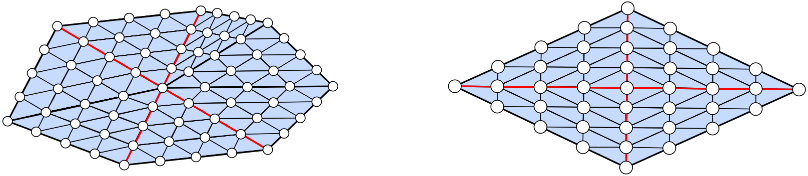
Theorem (2010)

Let Δ be a cell, and let smoothness $r \geq 1$. Suppose the number of different slopes $m \leq r + 2$. Then

$$S^r_{r+1}(\Delta) = S^r_{r+1}(\widetilde{\Delta}),$$

where $\widetilde{\Delta}$ is a cell obtained from Δ by removing the edges with no collinear counterparts.

Example: $r = 3$, $d = 4$, $m = 5$. Three black edges can be removed.



mixed derivatives

Theorem (2012)

Let Δ be a cell with no non-collinear and $2l$ collinear edges meeting at v. Then for any $s \in S_d^{l-1}(\Delta)$ any l-th order mixed derivative

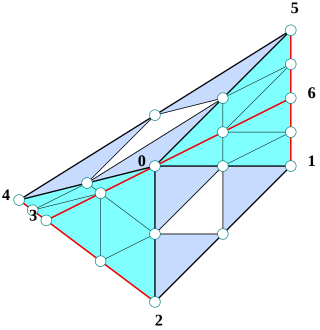
$$\frac{\partial^l s}{\partial u_{i_1} \cdots \partial u_{i_l}}(v),$$

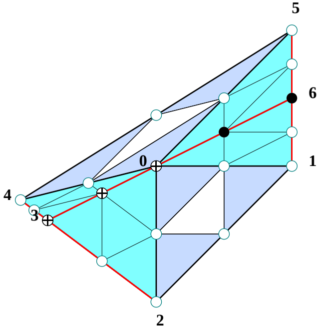
where $u_{i_1}, \ldots, u_{i_l}$ are pairwise distinct directions of non-collinear edges, exists.

one directional derivative

Theorem (2012)

Let $\triangle$ be a cell with four non-collinear edges meeting at the point v. Then there exists a unique straight line passing through v with the property that for any smooth quadratic spline s on $\triangle$, the restriction of s on this line is a univariate quadratic polynomial.





conclusions

- supersmoothness can help to compute and explain dimension
- supersmoothness could be a property of *every* multivariate spline
- the more symmetry the space has the less supersmoothness it possesses
- symmetry of both the partition and the smoothness functionals affects supersmoothness
- it appears that non-generic triangulations induce less supersmoothness
- what about really high values of n....