Connections of Wavelet Frames to Algebraic Geometry and Multidimensional Systems

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1. Construction of tight wavelet frames: UEP
2. Positivity vs. sum of squares (sos)
3. Connections to semi-definite programming
4. Connection to multidimensional systems
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Notations for Laurent polynomials:

$$\begin{aligned}
\mathbb{T}^d &= \{z \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\} \\
p &= \sum_{\alpha\in\mathbb{Z}^d} p_\alpha z^\alpha \in \mathbb{R}[\mathbb{T}^d] \\
p^* &= \sum_{\alpha\in\mathbb{Z}^d} p_\alpha z^{-\alpha}
\end{aligned}$$

Ex: $p(z_1, z_2) = 2^{-k-l-m}(1+z_1)^k(1+z_2)^l(1+z_1z_2)^m$

two-scale symbol of 3-directional box-spline

$$B(x) = 4\sum_\alpha p_\alpha B(2x-\alpha), \qquad x \in \mathbb{R}^2$$

$$\begin{aligned} M &\in \mathbb{Z}^{d\times d} && \text{scaling matrix} \\ G &= M^{-1}\mathbb{Z}^d/\mathbb{Z}^d \end{aligned}$$

defines a group action on $\mathbb{R}[\mathbb{T}^d]$:

$$p \mapsto p^\sigma(z_1,\dots,z_d) := p(e^{2\pi i\sigma_1}z_1,\dots,e^{2\pi i\sigma_d}z_d), \qquad \sigma \in G.$$

Ex: $M = 2I_2$

$$p^{(0,0)}(z_1,z_2) = 2^{-k-l-m}(1+z_1)^k(1+z_2)^l(1+z_1z_2)^m$$

$$p^{(1,0)}(z_1,z_2) = 2^{-k-l-m}(1-z_1)^k(1+z_2)^l(1-z_1z_2)^m$$

$$p^{(0,1)}(z_1,z_2) = 2^{-k-l-m}(1+z_1)^k(1-z_2)^l(1-z_1z_2)^m$$

$$p^{(1,1)}(z_1,z_2) = 2^{-k-l-m}(1-z_1)^k(1-z_2)^l(1+z_1z_2)^m$$

Unitary Extension Principle (Ron, Shen (1997)): Construction of tight wavelet frames

Let $p \in \mathbb{R}[\mathbb{T}^d]$, with $p(1,\ldots,1) = 1$, be the two-scale symbol of a refinable function $\phi \in L_2(\mathbb{R}^d)$. Find $q_j \in \mathbb{R}[\mathbb{T}^d]$, $1 \le j \le N$, such that

$$I - \left(p^\sigma\right)_{\sigma\in G} \left(p^\sigma\right)^*_{\sigma\in G} = \sum_{j=1}^{N} \left(q_j^\sigma\right)_{\sigma\in G} \left(q_j^\sigma\right)^*_{\sigma\in G}.$$

Then the functions

$$\psi_j(x) = \sum_{\alpha\in\mathbb{Z}^d} q_{j,\alpha}\phi(M^T x - \alpha), \qquad j = 1,\ldots,N,$$

generate a tight wavelet frame of $L_2(\mathbb{R}^d)$.

Questions for given $p \in \mathbb{R}[\mathbb{T}^d]$:

1. Do $q_1, \ldots, q_N \in \mathbb{R}[\mathbb{T}^d]$ exist?
2. What is the smallest number N (number of frame generators)?
3. What is the smallest degree of q_j's (support of frame generators)?

Find ways of construction or parameterization of all/some q_j's.

Background on UEP

- $I - (p^\sigma)(p^\sigma)^* = QQ^*$ implies the "sub-QMF" condition

$$f_p := 1 - \sum_{\sigma \in G} p^{\sigma *} p^\sigma \geq 0. \tag{1}$$

- Necessary and sufficient for the existence of q_j is the sum-of-squares (sos) decomposition

$$f_p = 1 - \sum_{\sigma \in G} p^{\sigma *} p^\sigma = \sum_{j=1}^{r} h_j^* h_j \tag{2}$$

with suitable $h_j \in \mathbb{R}[\mathbb{T}^d]$.

necessary: Cauchy-Binet formula for det QQ^*

sufficient: Lai, St. (2006) with G-invariant h_j, Charina et al. (2013)

Remark: Additional steps are required to pass from h_j in (2) to q_j in UEP.

Positivity vs. Sum of Squares

General result requires strict positivity:

- Schmüdgen's Positivstellensatz (1991):
  Let $g_1, \ldots, g_n \in \mathbb{R}[x_1, \ldots, x_d]$ and define
  $K := \{x \in \mathbb{R}^d : g_j(x) \geq 0, j = 1, \ldots, n\}$.

  If K is compact,
  then any $f \in \mathbb{R}[x_1, \ldots, x_d]$ with $f > 0$ on K can be written as

$$f = \sum_{\beta \in \{0,1\}^n} h_\beta \, g_1^{\beta_1} \cdots g_n^{\beta_n}, \qquad \text{with} \quad h_\beta \text{ sos.}$$

Does not apply to UEP : $f_p(1, \ldots, 1) = 0$

For non-negative $f \in \mathbb{R}[\mathbb{T}^d]$, the dimension d is crucial:

$d = 1$ Riesz-Fejer lemma:

$$f \geq 0 \iff f = h^* h \ \text{ with } \ h \in \mathbb{R}[\mathbb{T}] \qquad \text{(same degree)}$$

$d = 2$ Scheiderer's result in Manuscripta Math. 2006:

Let V be a non-singular affine variety over $\mathbb{R}$ of dimension 2, whose real points $V(\mathbb{R})$ are compact. Then every $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ is a sum of squares in $\mathbb{R}[V]$.

Ex: For 2-d butterfly scheme by Dyn, Gregory, Levin, we find $N = 13$ and degree(q_j) $\leq$degree(p).

$d \geq 3$

- There exists $f \in \mathbb{R}[\mathbb{T}^d]$ which is not sos

  Construction with homogeneous Motzkin polynomial in $\mathbb{R}[\mathbb{R}^3]$, which is

$$p(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$$

- For scaling matrix $M = 2I$, there exists $p \in \mathbb{R}[\mathbb{T}^d]$ with $p(1, \ldots, 1) = 1$ such that

$$f_p = 1 - \sum_{\sigma \in G} p^{\sigma *} p^{\sigma} \qquad \text{is not sos}$$

There are sufficient conditions also for $d \geq 3$.

- Scheiderer (2003): Let V be a nonsingular affine variety over $\mathbb{R}$ for which $V(\mathbb{R})$ is compact. If $f \geq 0$ on $V(\mathbb{R})$ and for every $\xi \in V(\mathbb{R})$ with $f(\xi) = 0$, the Hessian of f at ξ is positive definite, then f is a sum of squares in $\mathbb{R}[V]$.

Ex:

- If p is the two-scale symbol of a box-spline, f_p satisfies the condition on its Hessian; UEP constructions were known before, Gröchenig, Ron (1998), Chui, He (2001), Charina, St. (2008)

- The condition on the Hessian is not necessary:
For a 3-d interpolatory subdivision scheme by Chang et al. (2003), the function f_p has zero Hessian at some zero. We construct q_j's for UEP with $N = 31$.

Connections to semi-definite programming

1. Polynomials are written with the monomial vector $t(z) = (z^\alpha)_{\alpha \in I}$

$$p = t(z)^T \mathbf{p}, \qquad \mathbf{p} = (p_\alpha)_{\alpha \in I}$$

2. Due to $z^\alpha (z^\beta)^* = z^{\alpha - \beta}$ and $\sum_\alpha p_\alpha = 1$ we have

$$1 - pp^* = t(z)^T \Big(\underbrace{\operatorname{diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}^T}_{=: R} \Big) t(z^*)$$

R is called a Gram-matrix of $1 - pp^*$.

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3. Find a symmetric matrix $S \in \mathbb{R}^{|I| \times |I|}$ such that

$$R + S \quad \text{is positive semi-definite}$$

   and

$$\sum_{\alpha \in I} S_{\alpha, \alpha+\beta} = 0 \quad \text{for all} \quad \beta.$$

4. By

$$1 - pp^* = t(z)^T \Big(\underbrace{R + S}_{\text{semidef.}} \Big) t(z^*),$$

any decomposition $R + S = \sum_{j=1}^{N} \mathbf{h}_j \mathbf{h}_j^T$ gives polynomials $h_j = t(z)^T \mathbf{h}_j$ with

$$1 - pp^* = \sum_{j=1}^{N} h_j h_j^*.$$

Note: Semi-definiteness of $R + S$ requires extra care in SDP standard routines.

By the "sum rules"

$$\frac{1}{|\det M|} = \sum_{\beta} p_{\gamma+M^T\beta}, \qquad \gamma \in \mathbb{Z}^d/M^T\mathbb{Z}^d,$$

we can obtain solutions q_j to UEP by stronger constraints:

3'. Find a symmetric matrix $S \in \mathbb{R}^{|I|\times|I|}$ such that

$$R + S \quad \text{is positive semi-definite}$$

and

$$\sum_{\alpha \in I\cap(\gamma+M^T\mathbb{Z}^d)} S_{\alpha,\alpha+\beta} = 0 \quad \text{for all} \quad \beta, \;\; \gamma \in \mathbb{Z}^d/M^T\mathbb{Z}^d.$$

4. $R + S = \sum_{j=1}^{N} \mathbf{q}_j \mathbf{q}_j^T$ gives polynomials $q_j = t(z)^T \mathbf{q}_j$ with

$$I - (p^\sigma)(p^\sigma)^* = \sum_{j=1}^{N} (q_j^\sigma)(q_j^\sigma)^*.$$

Connection to multidimensional systems

Let p be a polynomial, $\mathbb{D}^d = \{|z_1| < 1, \ldots, |z_d| < 1\}$ the open polydisk in $\mathbb{C}^d$, and

$$|p(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}^d.$$

Results by Agler (1990), Ball, Trent (1998), Agler, McCarthy (1999):

The following are equivalent:

(a) p satisfies a von Neumann inequality; i.e., for every family $T_1, \ldots, T_d \in \mathcal{L}(H)$ of commuting contractions on a Hilbert space H,

$$\|p(T_1, \ldots, T_d)\|_{\mathrm{op}} \leq 1.$$

(b) There exist $n_1, \ldots, n_d \in \mathbb{N}$ and a matrix $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{(1+N)\times(1+N)}$, with $N = \sum_j n_j$ and $I - V^* V \geq 0$, such that

$$p(z) = A + BE(z)\left(I - DE(z)\right)^{-1} C,$$

where $E(z) = \begin{pmatrix} z_1 I_{n_1} & & \\ & \ddots & \\ & & z_d I_{n_d} \end{pmatrix}.$

The matrix $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{(1+N)\times(1+N)}$ is called a *transfer function realization* for p.

To obtain an sos-decomposition of $1 - |p|^2$:

- Take $I - V^*V = X^*X$, with $X = [Q, Y] \in \mathbb{R}^{n_0\times(1+N)}$ and first column Q.
- Then $\begin{pmatrix} Q & Y \\ A & B \\ C & D \end{pmatrix}$ is an isometry,
  the polynomial vector

$$q(z) = Q + YE(z)\left(I - DE(z)\right)^{-1} C$$

  gives

$$1 - |p(z)|^2 = \sum_{j=1}^{n_0} |q_j(z)|^2, \qquad z \in \mathbb{D}^d.$$

Application to UEP requires:

- operator version of the transfer "function" realization to vectors $(p^\sigma(z))_{\sigma\in G}$
- extension of the sub-QMF condition to the polydisk:

$$1 - \sum_{\sigma\in G} |p^\sigma(z)|^2 \geq 0 \quad \text{for all} \quad z \in \mathbb{D}^d.$$

In return, we obtain a parameterization of families of frame generators, and of suitable two-scale symbols p.

Results and algorithms:

- $d = 1$: system theory is completely developed
- $d = 2$: every 2-d polynomial p with $|p|^2 \leq 1$ on the polydisk has a transfer function realization (consequence of Ando's dilation theorem)

  Algorithm by Kummert (1989)
- $d \geq 3$: examples of polynomials which do not have a transfer function realization, (Varopoulas)

Conclusion

UEP construction of tight wavelet frames

- is closely connected with sos-decomposition of non-negative trigonometric polynomials,
- profits from recent results in real algebraic geometry and multidimensional systems,
- can be automated by semi-definite programming or transfer function representation.