On Approximate Polynomials,

By

Sōichi Kakeya in Sendai.

Lately, Mr. J. P. 41 has proved the following interesting theorem: (1) Let \( f(x) \) be a continuous function of a real variable \( x \) in the interval \( 0 \leq |x| \leq a < 1 \), which vanishes at the point \( x = 0 \), and let \( \varepsilon \) be an arbitrary positive number, then there exists a polynomial \( P(x) \) with integral coefficients such that

\[
|f(x) - P(x)| < \varepsilon,
\]

for all values of \( x \) in the interval \( 0 \leq |x| \leq a \).

In his theorem, it is necessary that the number \( a \), which is the upper limit of \( |x| \), is less than unity. To extend the theorem to the case when \( a \) is equal to unity is the aim of the following lines.

1. For our purpose, it is necessary to introduce a certain new condition for the given function \( f(x) \); and the theorem thus extended runs as follows:

Let a function \( f(x) \) be continuous in the interval \( 0 \leq |x| \leq 1 \) and

\[
f(0) = f(1) = f(-1) = 0,
\]

then, for any given positive number \( \varepsilon \), there exists a corresponding polynomial \( P(x) \) with integral coefficients such that

\[
|f(x) - P(x)| < \varepsilon
\]

for all values of \( x \) in the interval \( 0 \leq |x| \leq 1 \).

To prove this theorem, we first consider an auxiliary polynomial

\[
y = x(x+1)(x-1).
\]

As it is easily seen, the new variable \( y \) varies monotonously from 0 to \( \frac{2}{3\sqrt{3}} \), while \( x \) varies from \( -1 \) to \( -\frac{1}{\sqrt{3}} \), and \( y \) varies monotonously from \( \frac{2}{3\sqrt{3}} \) to 0, while \( x \) varies from \( -\frac{1}{\sqrt{3}} \) to 0. Consequently the two values \( p \) and \( q \) of \( x \) such that

\[
\begin{align*}
\sqrt{1 - q} &= \frac{2}{3\sqrt{3}} \\
p &= -\frac{1}{\sqrt{3}} \\
q &= \frac{2}{3\sqrt{3}}
\end{align*}
\]

\[(1)\] Tôhoku Math. Jour. vol. 6, 1914, p. 42.
(1) \(\psi(a) = 0\) for \(1 - g \leq |a| \leq 1 - \gamma\),
(2) \(\psi(a) = 1\) for \(\frac{1}{\sqrt{3}} \leq |a| \leq 1 - g\),
(3) \(\psi(a) = 1\) for \(\frac{1}{\sqrt{3}} \leq |a| \leq 1\).

Then the form of \(\psi(a)\) is determined as follows:

- For \(|a| < 1\), \(\psi(a) = 0\) or \(\psi(a) = 1\) depending on the value of \(2\sqrt{3} - 3\) or \(2\sqrt{3} - 2\).

Let it be denoted by

\(\psi(a) = \phi(a)\).

Therefore, if a function \(\phi(a)\) is continuous in the interval \((-1, 1)\), it varies from 0 to 1 in the intervals \(1 - g \leq |a| \leq 1\) and corresponding two variables. When \(a\) varies from 0 to 1, the variation of \(\phi(a)\) is symmetric with respect to the interval \((-1, 1)\).

Consequently, we must have

\(\phi(a) = \psi(a)\).

(3) \(\psi(a) = \phi(a)\) for \(0 \leq |a| \leq \frac{1}{\sqrt{3}}\).
point \( y = 0 \). Consequently, by the theorem of Mr. Pail, we can find a polynomial \( Q(y) \) with integral coefficients such that
\[
| \phi(y) - Q(y) | < \varepsilon_1 \quad \text{for} \quad 0 \leq |y| \leq \frac{2}{3\sqrt{3}}. \tag{8}
\]
If we put
\[
Q(y) = R(x),
\]
\( R(x) \) is also a polynomial with integral coefficients and is such that
\[
| \phi(x) - R(x) | < \varepsilon_1 \quad \text{for} \quad 0 \leq |x| \leq 1. \tag{10}
\]
Again, by the same theorem, we can find a polynomial \( S(x) \) with integral coefficients such that
\[
| f(x) - S(x) | < \varepsilon_1 \quad \text{for} \quad 0 \leq |x| \leq 1 - q_1. \tag{11}
\]
From (10) and (11), we get
\[
| f(x) \phi(x) - S(x) R(x) | < | f(x) | \varepsilon_1 + | \phi(x) | \varepsilon_1 + \varepsilon_1 \varepsilon_2 < M \varepsilon_1 + \varepsilon_1 \varepsilon_2, \tag{12}
\]
for the interval \( 0 \leq |x| \leq 1 - q_1 \), where \( M \) is the greatest magnitude of \( |f(x)| \) in the interval \((-1, 1)\). Specially, if we consider only the interval in which \( \phi(x) \) becomes 1, we get
\[
| f(x) - S(x) R(x) | < M \varepsilon_1 + \varepsilon_1 + \varepsilon_1 \varepsilon_2 \quad \text{for} \quad p_1 \leq |x| \leq 1 - q_1. \tag{13}
\]
Since, in the intervals \( p_1 \leq |x| \leq p_1 \) and \( 1 - q_1 \leq |x| \leq 1 - q_1 \), \( \phi(x) \) varies monotonously from 1 to 0, we have
\[
| f(x) - S(x) R(x) | \leq | f(x) \phi(x) - S(x) R(x) | + | f(x) - f(x) \phi(x) | < M (p_1 + p_1) + M \varepsilon_1 + \varepsilon_1 + \varepsilon_1 \varepsilon_2 \tag{14}
\]
for \( p_1 \leq |x| \leq p_1 \) or \( 1 - q_1 \leq |x| \leq 1 - q_1 \), where \( M (p_1, p_1) \) is the greatest magnitude of \( |f(x)| \) in the intervals of (14).

In the remaining intervals \( \phi(x) \) becomes zero and hence \( |R(x)| \) becomes less than \( \varepsilon_1 \), so we have
\[
| f(x) - S(x) R(x) | \leq | f(x) | + | S(x) | + | R(x) | < M (p_1) + N (p_1) \varepsilon_1 \quad \text{for} \quad 0 \leq |x| \leq p_1, \quad \text{or} \quad 1 - q_1 \leq |x| \leq 1, \tag{15}
\]
where \( M (p_1) \) and \( N (p_1) \) are the greatest magnitudes of \( |f(x)| \) and \( |S(x)| \) respectively in the intervals of (15).

Take \( p_i, M (p_1, p_1) \) as function vanishing small, then \( M \varepsilon_1 \) an hand member sufficiently sma

for all values be supposed
If we p

\( P(x) \) is also get

for all values proved.

2. In \( f(x) \) vanishing by the \( \alpha = 0, 1 \) and

function

\( g(x) = f(x) \)

vanishes at

integral convention.

The abs

\( f(-1) \) can \( P(-1) \) resp

for sufficient

must be even.

The next

(1) This
Take $p_1$ and $p_2$ in the above discussion sufficiently small, then $M(p_1, p_2)$ and $M(p_2)$ become sufficiently small, for $f(x)$ is a continuous function vanishing at the points 0, 1, and $-1$. Next take $\varepsilon$, sufficiently small, then the quantity $N(p_2)$ is determined. Lastly, take $\varepsilon$, so small that $M(p_2)$ and $N(p_2)$ also become sufficiently small. Then the right hand members of all the inequalities (13), (14) and (15) become sufficiently small. Hence, combining these three inequalities, we get

$$|f(x) - S(x) R(x)| < \varepsilon$$  \hspace{1cm} (16)

for all values of $x$ in the combined interval $0 \leq |x| \leq 1$, where $\varepsilon$ can be supposed to be an arbitrarily small number.

If we put

$$S(x) R(x) = P(x),$$  \hspace{1cm} (17)

$P(x)$ is also a polynomial with integral coefficients, and, from (16) we get

$$|f(x) - P(x)| < \varepsilon$$  \hspace{1cm} (18)

for all values of $x$ in the interval $0 \leq |x| \leq 1$. Thus our theorem is proved.

2. In the preceding theorem, we have given the condition that $f(x)$ vanishes at the points 0, 1, and $-1$. This condition can be replaced by the condition that $f(x)$ takes such the integral values at the points 0, 1 and $-1$ that $f(1) + f(-1)$ is even. For, in such a case, the function

$$g(x) = f(x) - \left[ f(0) + \frac{f(1) - f(-1)}{2} - 2f(0)|x| \right]$$

vanishes at the said three points and $g(x) - f(x)$ is a polynomial with integral coefficients.

The above new condition is also necessary. For, since $f(0), f(1), f(-1)$ can be approached indefinitely near by the integers $P(0), P(1), P(-1)$ respectively, they must be also integers and

$$f(0) = P(0), \quad f(1) = P(1), \quad f(-1) = P(-1),$$

for sufficiently small $\varepsilon$. That

$$f(1) + f(-1) = P(1) + P(-1)$$

must be even is a special consequence of the following general theorem: (1)

The necessary and sufficient condition that the integral values $u_1, u_2,$

\hspace{1cm} (1') This follows at once from Newton's formula of interpolation.
......, $u_n$ can be attained by a polynomial $P(x)$ with integral coefficients, for the integral values $a_1, a_2, \ldots, a_n$ of $x$, is that all of the $n-1$ expressions

$$
\frac{u_1}{(a_1-a_2)(a_1-a_3) \ldots (a_1-a_k)} + \frac{u_2}{(a_2-a_1)(a_2-a_3) \ldots (a_2-a_k)} + \ldots
$$

$$
+ \frac{u_k}{(a_k-a_1)(a_k-a_2) \ldots (a_k-a_{k-1})}
$$

$k=2, 3, \ldots, n$

should be integers.

To extend the theorem to an interval greater than or equal to $(-2, 2)$ is impossible, unless the function $f(x)$ itself is a polynomial in that interval. For if there are two different polynomials $P_1(x)$ and $P_2(x)$ with integral coefficients such that

$$|f(x) - P_1(x)| < 1, \quad |f(x) - P_2(x)| < 1, \quad P_1(x) - P_2(x) \neq \text{const.}$$

in the interval $0 \leq |x| \leq a$ $(a \geq 2)$, then we get a polynomial

$$P_1(x) - P_2(x) = C_0 x^n + C_1 x^{n-1} + \ldots + C_n \quad (C_0 \neq 0)$$

with integral coefficients such that

$$|C_0 x^n + C_1 x^{n-1} + \ldots + C_n| < 2 \quad \text{for} \quad 0 \leq |x| \leq a;$$

and this contradicts the known theorem of Tschebyscheff (1) that there exists at least one point $x$ in the interval $(-a, a)$ for which

$$|C_0 x^n + C_1 x^{n-1} + \ldots + C_n| \geq \frac{C_0}{2^{a-1}} a^n \geq 2C_0.$$

I cannot yet find out the upper limit of the intervals to which the theorem can be extended.

(1) Oeuvres, t. 1, pp. 273-278.