ON AN EXTREMAL PROPERTY OF CHEBYSHEV POLYNOMIALS

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Given a closed interval $S = [a, b]$ of length $\ell = b - a$, and two positive numbers $\lambda = \theta \ell$, $0 < \theta < 1$, and $0 < \kappa$, we consider the following problem:

**Problem.** Find an exact upper bound on the quantity

$$\max_{a \leq x \leq b} |P_n(x)|$$

where $P_n$ is a polynomial of degree at most $n$ satisfying the inequality

$$|P_n(x)| \leq \kappa$$

on a set of points (otherwise undetermined) $E \subset S$ of measure $\geq \lambda$.

We will show that the upper bound in question has the exact value

$$M = \kappa T_n \left( \frac{2\ell}{\lambda} - 1 \right) = \kappa T_n \left( \frac{2\theta}{\theta - 1} \right)$$

where $T_n$ is the trigonometric polynomial of degree $n$

$$T_n(z) = \frac{1}{2} \left\{ (z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n \right\}.$$  

**Solution.** We will first verify that (1) attains the value (3) for the two Chebyshev polynomials

$$P_{n,1}(x) = \kappa T_n \left( \frac{2x - a - (a + \lambda)}{\lambda} \right)$$

and

$$P_{n,2}(x) = \kappa T_n \left( \frac{2x - (b - \lambda) - b}{\lambda} \right),$$

1The author encountered this problem in the course of his work on the convergence of a certain process of successive approximations that he had proposed for the effective calculation of the polynomial of best approximation to a bounded function $f(x)$ on a uniformly bounded set of points. (Cf. my note, Comptes Rendus, Paris, 30, VII, 1934 and also my article in Ukrainian: “On the methods for realizing the best approximation of functions in the sense of Chebyshev”, Acad. des Sc. d’Ukraine, 1935, pp. 99–100).
which satisfy the condition (2), one on the interval \([a, a + \lambda]\), the other on the interval \([b - \lambda, b]\). It remains to prove that among all admissible polynomials \(P_n(x)\), the two polynomials (5) and (6) are the only (up to multiplication by \(\pm 1\)), for which the quantity (1) attains the value (3).

Let \(P_n(x)\) be an admissible polynomial different from (5) and (6); let \(E \subset S\) be a set of points on which (2) holds. This set of points is evidently composed of a certain number \(\nu \leq n\) of closed intervals some which can be one point. Let

\[
\sigma_1 = [\alpha_1, \beta_1], \quad \sigma_2 = [\alpha_2, \beta_2], \ldots, \sigma_m = [\alpha_m, \beta_m]
\]

be those of them \((m \leq \nu)\) of positive length arranged in increasing order. Let \(\xi \in S\) be a point such that \(|P_n(x)|\) attains its maximum value on the interval \([a, b]\):

\[
|P_n(\xi)| = \max_{a \leq x \leq b} |P_n(x)|. \tag{8}
\]

We will show that \(|P_n(\xi)| \leq M\), where \(M\) designates the value (3).

We distinguish between three cases depending on:

\[
\xi > \beta_m, \quad \xi < \alpha_1 \text{ or finally } \beta_i < \xi < \alpha_{i+1}
\]

where in the last case \(i \in \{1, 2, \ldots, m - 1\}\).

We start by considering the first case. Let \(x_1 = a, x_2, x_3, \ldots, x_{n+1} = a + \lambda\) be the points on the interval \([a, a + \lambda]\), where the Chebyshev polynomial (5) attains, with alternating sign, the values \(\pm \kappa\). Let, in addition, \(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}\) be the \(n + 1\) points that we take in the set \(E\) satisfying the following conditions: firstly, \(\bar{x}_1 = \alpha_1\); then for \(\ell = 2, 3, \ldots, n + 1\) let \(\bar{x}_\ell\) be the first of the points in \(E\) (traversing this set of points from left to right) for which

\[
\text{mes}(\bar{x}_1, \bar{x}_\ell) \cdot E = x_\ell - x_1, \tag{10}
\]

the product in the parenthesis meaning the set of points appearing both in the interval \([\bar{x}_1, \bar{x}_\ell]\) and in the set \(E\). (Trans: The intersection of the two sets.)

Applying the Lagrange interpolation formula, one time with the polynomial (5) and another time with the polynomial \(P_n(x)\), we can write the following two equalities:

\[
M = P_{n,1}(b) = \sum_{i=1}^{n+1} \frac{(b - x_1) \cdots (b - x_{i-1})(b - x_{i+1}) \cdots (b - x_{n+1})}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_{n+1})} P_{n,1}(x_i) \tag{11}
\]

\[
P_n(\xi) = \sum_{i=1}^{n+1} \frac{(\xi - \bar{x}_1) \cdots (\xi - \bar{x}_{i-1})(\xi - \bar{x}_{i+1}) \cdots (\xi - \bar{x}_{n+1})}{(\bar{x}_i - \bar{x}_1) \cdots (\bar{x}_i - \bar{x}_{i-1})(\bar{x}_i - \bar{x}_{i+1}) \cdots (\bar{x}_i - \bar{x}_{n+1})} P_n(\bar{x}_i). \tag{12}
\]

On comparing their right-hand parts term by term, one notes the following relations:

\[
\begin{align*}
\alpha) & \quad |P_{n,1}(x_i)| = \kappa; \quad |P_{n,1}(\bar{x}_i)| \leq \kappa \\
\beta) & \quad b - x_j \geq \xi - \bar{x}_j \geq 0 \\
\gamma) & \quad |x_i - x_j| \leq |\bar{x}_i - \bar{x}_j|, \quad i, j = 1, 2, \ldots, n + 1; \quad j \neq i.
\end{align*}
\]
Moreover, one also sees that the $n + 1$ terms on the last part of (11) are all the same sign (being +), which need not hold in (12). Thus one also has

$$|P_n(\xi)| < M,$$

at least that $P_n(x)$ is not identical to $\pm P_{n,1}(x)$.

In the second case (9), that is to say when $\xi < \alpha_1$, the reasoning is totally analogous, on replacing the polynomial (5) by (6).

Finally in the case in (9)

$$\beta_i < \xi < \alpha_{i+1}$$

set

$$\frac{\text{mes}([a, \xi] \cdot E)}{\xi - a} = \theta_1, \quad (15)$$

$$\frac{\text{mes}([\xi, b] \cdot E)}{b - \xi} = \theta_2. \quad (16)$$

It is clear that the two numbers $\theta_1$ and $\theta_2$ can not be at the same time less than $\theta = \frac{\lambda}{T}$. Replacing, in the previous reasoning, the interval $[a, b]$ once by $[a, \xi]$ and another time by $[\xi, b]$, one has simultaneously

$$|P_n(\xi)| < \kappa T_n \left( \frac{2}{\theta_1} - 1 \right)$$

$$|P_n(\xi)| < \kappa T_n \left( \frac{2}{\theta_2} - 1 \right), \quad (17)$$

and one of the right hand sides above is certainly $\leq M$ and the proof is complete.

We have simultaneously obtained a simple proof of a known theorem of Chebyshev\(^2\) which derives from our reasoning when a priori restricting the field of admissible polynomials\(^3\).

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\(^3\)As understood, one sets a priori $E = [a, a + \lambda]$ or $E = [b - \lambda, b]$. 