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Dear Phil:

Lately I have been busy with Euclidean polyhedra from the point of view of distance geometry, but on and off I return to ortho
genics. The other day I noticed that the triangle formulae
in the simplex plane generalize nicely to higher order derivatives.
The source is again the Hermite-Geronimo formula

\[ f(x_0, x_1, \ldots, x_n) = \int_{\Delta_n} \cdots \int_{\Delta_1} f^{(n)}(x_0t_0 + x_1t_1 + \cdots + x_nt_n) \, dt_1 \cdots dt_n \]

where \( t_0 = 1 - t_1 - \cdots - t_n \) and where the integration is to be carried out over the simplex \( \Delta_n : t_1 \geq 0, \ldots, t_n \geq 0, \sum t_i \leq 1 \).

A projection onto the real axis allows us to show that the kernel \( M(x; x_0, x_1, \ldots, x_n) \) in the fundamental formula

\[ f(x_0, x_1, \ldots, x_n) = \frac{1}{n!} \int_{\Delta_n} f^{(n)}(x) \, M(x; x_0, \ldots, x_n) \, dx \]

may be interpreted as follows: Interpret the \( x_0 \)-axis as one of the coordinate axes of \( n \)-dimensional space. Treat at \( x_0, x_1, \ldots, x_n \) \n hypersurface orthogonal to the \( x \)-axis at these points and select at will in each of these hyperplanes a point, which
the points that the simplex having these points as vertices, should have \( n \)-dimensional volume unity. If we project
the volume of this simplex on the \( x \)-axis we obtain the \( \ell \)-th mean density function \( M(x; x_0, \ldots, x_n) \) appearing in (2).

I now turn to the complex domain. Let \( z_0, z_1, \ldots, z_n \)
be distinct points of the complex plane and let \( f(z) \) be
regular in the convex hull \( \Pi \) of these points. Again we
cover at \( z_0, z, \ldots, z_n \) in the orthogonal complements
of the plane (these are of dimension \( n-2 \)) and select in each of them a
point so that the simplex has \( \text{det}(\Delta) \) volume unity.
We now project the volume of this simplex onto the plane
and denote the surface density function by
\[
M(x, y; z_0, \ldots, z_n).
\]
Then the following formula holds
\[
\Psi(\tilde{z}_0, \ldots, \tilde{z}_n) = \frac{1}{n!} \int_\Pi f^{(n)}(z) M(x, y; z_0, \ldots, z_n) \, dx \, dy
\]
For \( n = 2 \) this gives the old triangle formula
\[
\Psi(z_0, z_1, z_2) = \frac{1}{2A} \int_T f^{(2)}(z) \, dx \, dy
\]
where \( A \) is the area of the triangle \( T \) of vertices \( z_0, z_1, z_2 \).

Joining all pairs of points \( z_j, z_k \) \( (j < k) \) by segments,
the polygon \( \Pi \) is dissected into disjoint polygons in each of
which \( M(x, y) \) is a polynomial in \( x \) and \( y \) of joint degree
\( n-2 \), while \( M(x, y) = 0 \) outside \( \Pi \). Moreover, the function
\( M(x, y) \) has continuous partial derivatives of all orders \( \leq n-3 \).

Moreover, these properties always determine \( M(x, y) \) uniquely
up to a constant factor. Another property is this: \( M(x, y) \)
inside \( \Pi \) and
\[
\text{det} M(x, y) \quad (x, y \text{ inside } \Pi)
\]
is a concave function. In particular, \( M(x, y) \) has exactly
one maximum point.

\( X \) These assume that the points \( z_0, \ldots, z_n \) are in \textit{general position},
i.e. no three are collinear.
Thus for \( n = 3 \) the surface \( \mathbb{Z} = M(x,y) \) is a pyramid.

In case all the points \( z_0, z_1, \ldots, z_n \) are on the boundary of the polygon \( \Gamma \), then \( M(x,y) \) can be represented inside \( \Gamma \) by means of the truncated Heav function \( x_n \).

Here is an example: For \( n = 4 \) and
\[
M(x,y) = 2, \quad z_1 = 1 + i, \quad z_2 = -1 + i, \quad z_3 = 1 - i, \quad z_4 = 1 - i,
\]
\( M(x,y) \) is up to a positive factor, which I did not determine, identical inside \( \Gamma \) to
\[
2 + 4x + 4x^2 - 6y + 6(x-1)^2 + 6(y-2)^2 + 3(y-x)^2
\]
\[+ (x-3y-2)^2 + 3(-y-x)^2.\]

I am very much interested in these 2-dimensional frequency functions
\( M(x,y; z_0, \ldots, z_n) \)
because I suspect that the limits of such frequency functions (with the \( z_i \) depending on \( n \) and as \( n \to \infty \)) in the usual sense of probability theory, ought to be interesting frequency functions.

The similar question on the line lead to the Polya frequency functions. What will one get in the plane, if anything?

I hope these remarks did not bore you.

With greetings and good wishes.

Yours,

F.J. Schenkenberg

I add a sketch of the second degree \( M(x,y) \) (\( n = 4 \))
for the case when \( z_0, \ldots, z_4 \) are the vertices of a regular pentagon. All vertical sections are, of course, ordinary sine functions.