In the discussion of the error in linear interpolation at the vertices of a simplex, certain simple geometric facts concerning spheres containing some or all of the vertices of a simplex play a role.

If two spheres,

$$S_r(c_r) := \partial B_r(c_r), \qquad S_s(c_s) := \partial B_s(c_s),$$

have the pointset V in common, then $\langle c_r - c_s, \cdot \rangle$ is constant on V, hence on affine(V), hence $c_r - c_s$ must be perpendicular to affine(V). In particular, if $S_r(c_r)$ is the sphere containing all the vertices of the simplex (i.e., the circumscribed sphere), while $S_s(c_s)$ is the sphere of smallest radius containing the simplex, then c_s is in the relative interior of the possibly lower order simplex spanned by the set V of vertices it contains from the simplex, and, as $c_r - c_s$ is perpendicular to affine(V) while, for any $v \in V$, $v - c_s$ is in affine(V), Pythagoras tells us that

$$\|c_r - c_s\|^2 = \|c_r - v\|^2 - \|v - c_s\|^2 = r_r^2 - r_s^2.$$

Shayne Waldron points out the following immediate consequence of this identity: Since

$$\|x - c_r\|^2 = \|x - c_s\|^2 + 2\langle x - c_s, c_s - c_r \rangle + \|c_s - c_r\|^2,$$

therefore

$$r_r^2 - \|x - c_r\|^2 = r_s^2 - \|x - c_s\|^2 \; - \; 2\langle x - c_s, c_r - c_s \rangle.$$

In particular,

$$r_r^2 - \|\cdot - c_r\|^2 = r_s^2 - \|\cdot - c_s\|^2 \quad \text{ on affine}(V).$$

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