B-spline convolutions

Recall that

$$M(\cdot|t_0,\ldots,t_k) : x \mapsto k\boldsymbol{\Delta}(t_0,\ldots,t_k)(x-\cdot)_+^{k-1}$$

is the B-spline with knot sequence $t_0,\ldots,t_k$ that integrates to 1; it is nonnegative and has its support in $[t_0 \mathinner{\ldotp\ldotp} t_k]$; it is the Peano kernel for the corresponding divided difference in the sense that

$$\boldsymbol{\Delta}(t_0,\ldots,t_k)f = \int M(\cdot|t_0,\ldots,t_k)D^k f/k!$$

for all smooth enough f. Hence, the convolution

$$C := M(\cdot|t_0,\ldots,t_k) * M(\cdot|s_0,\ldots,s_h) : x \mapsto \int M(x-y|t_0,\ldots,t_k)M(y|s_0,\ldots,s_h)\,\mathrm{d}y$$

of two B-splines is a nonnegative function with support in

$$[t_0 \mathinner{\ldotp\ldotp} t_k] + [s_0 \mathinner{\ldotp\ldotp} s_h]$$

and is piecewise polynomial of order $k+h$ since it is of the form

$$x \mapsto \sum_{i=0}^{h} a(i) \sum_{j=0}^{k} b(j)(x - s_i - t_j)_+^{k+h-1},$$

and this also shows its knot sequence to comprise the entries

$$\{s_i + t_j : i = 0{:}r, j = 0{:}k\}.$$

If $s_i = i$ and $t_j = j$ for all relevant i and j, then C has knots only at $0, 1, \ldots, k+h$, and since, as the convolution product of two functions that integrate to 1, it integrates to 1, it must necessarily be the cardinal B-spline $M(\cdot|0{:}(k+h))$. But for general t and s, it need not be a B-spline, however may well have nonnegative B-spline coefficients with respect to its knot sequence.

An early reference on integrals involving B-splines is BoorLycheSchumaker76; see also Neuman81b and the page `www.cs.wisc.edu/∼deboor/toast/pages004.html` on “Moments and Fourier transform of a B-spline”.