The Bernstein basis

In anticipation of the multivariate setup, we define here the elements of the Bernstein bases by

$$b_{j,k}(x) := \binom{j+k}{j} x^j (1-x)^k, \quad j,k = 0,1,2,\ldots.$$

Then the $(n+1)$-sequence $\mathbf{b}_n := (b_{j,n-j} : j = 0{:}n)$ is in the $(n+1)$-dimensional linear space $\Pi_{\le n}$ and is linearly independent since, e.g., the matrix

$$(D^i b_{j,n-j}(0) : i,j = 0{:}n)$$

is triangular with nonzero diagonal entries, hence invertible, therefore a basis for $\Pi_{\le n}$.

Note that

$$\sum_j b_{j,n-j}(x) = \sum_j \binom{n}{j} x^j (1-x)^{n-j} = (x \;+\; (1-x))^n = 1.$$

Hence, for any $k \le n$,

$$\begin{aligned} x^k &= x^k \sum_{j=0}^{n-k} \binom{n-k}{j} x^j (1-x)^{n-k-j} \\ &= \sum_{j=k}^{n} \binom{n-k}{j-k} x^j (1-x)^{n-j} \\ &= \sum_{j=0}^{n} c(k,n,j) b_{j,n-j}(x) \end{aligned}$$

with

$$c(n,k,j) := \binom{n-k}{j-k} \Big/ \binom{n}{j} = \frac{(n-k)!j!(n-j)!}{(n-j)!(j-k)!n!} = \frac{j(j-1)\cdots(j-k+1)}{n(n-1)\cdots(n-k+1)},$$

using the fact that $c(n,k,j)$ so defined is zero for $j < k$ since $\binom{n-k}{j-k}$ is then zero.

It follows that, for an arbitrary $p \in \Pi_{\le n}$,

$$p = \sum_{j=0}^{n} \widehat{p}_{n,j} b_{j,n-j},$$

with

$$\widehat{p}_{n,j} := \sum_{i=0}^{j} \binom{n-i}{j-i} \Big/ \binom{n}{j} D^i p(0)/i!.$$

For the multivariate Bernstein basis, see `(multivariate) polynomials > polynomial forms > BB-form`.