The Bernoulli monospline

Let $(p_k : k = 0, 1, 2, \ldots)$ be the Appell sequence for the linear functional

$$\lambda : f \mapsto \int_0^1 f = \Delta(0,1)D^{-1}f.$$

This means that, for each k, p_k is the unique polynomial of degree $\le k$ for which

$$\lambda D^j p_k = \delta_{jk}, \quad \text{all} \;\; j,$$

hence

$$Dp_k = p_{k-1}, \text{all} \;\; k,$$

and therefore

$$p_k = ()^k/k! + \text{l.o.t.},$$

as well as

$$p_k = q_k - \lambda q_k, q_k := \int_0^{\cdot} p_{k-1}, \quad k = 1, 2, \ldots.$$

In particular, for our particular λ,

$$p_k = \begin{cases} ()^0, & k = 0; \\ ()^1 - 1/2, & k = 1; \\ ()^2/2 - ()^1/2 + 1/12, & k = 2; \\ ()^3/6 - ()^2/4 + ()^1/12, & k = 3; \\ ()^4/24 - ()^3/12 + ()^2/24 - 1/720, & k = 4. \end{cases}$$

Moreover,

$$D^j p_k(0) = D^j p_k(1), \quad j = -1, 0, \ldots, k-2.$$

It follows that the 1-periodic extension

$$\mathcal{B}_k(t) := p_k(t - \lfloor t \rfloor), \quad t \in \mathbb{R},$$

of p_k to all of $\mathbb{R}$ is piecewise kth degree with breaks only at the integers and in $C^{k-2}(\mathbb{R})$. This makes it a monospline of degree k with simple knots, at the integers. It is called the **Bernoulli monospline of degree** k, and has appeared in various contexts in Approximation Theory, due to the following.

It can be shown (see, e.g., pp. 120–121 of BojanovHakopianSahakian93) that

$$k!\mathcal{B}_k(2\pi t) = D_k(t) := \sum_{r=1}^{\infty} \frac{\cos(rt - k\pi/2)}{r^k},$$

with

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f + \frac{1}{\pi}\int_0^{2\pi} D_k(x-s)D^k f(s)\,\mathrm{d}s$$

holding for $x \in \mathbb{R}$ and for any 2π-periodic $f \in C^{(k-1)}(\mathbb{R})$ with $D^k f$ locally in L_1.

Further, since $\lambda p_k = \int_0^1 p_k = 0$ for $k > 0$, each p_k, $k > 0$, has at least one zero in $(0\,..\,1)$. Further, since λ is invariant under the change of variable

$$Tf(t) := f(1-t), \quad t \in \mathbb{R},$$

the uniqueness of the corresponding Appell polynomials implies that p_k is even or odd around $t = 1/2$ according to whether k is even or odd. Since also (at least for $k > 1$), $p_k(0) = p_k(1)$, this implies that $\mathcal{B}_k$ has zeros at the integers when k is odd, while it has to have at least 2 zeros (counting multiplicity) in $(0\,..\,1)$ when k is even. This implies that, regardless of whether k is even or odd, $\mathcal{B}_k$ has at least two zeros in each half-open interval $[0\,..\,1)$. Could there be more? No, not by Micchelli's bound on the number of zeros of a monospline.

06mar04