Chapter II. Splines with uniform knots

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1. Specialization to the knot sequence $\mathbb{Z}$.

This chapter deals with splines with a uniform knot sequence, i.e., with the knot sequence

$$\mathbb{Z} \;:=\; (j)_{j=-\infty}^{\infty}$$

and, more generally, the sequence $h\mathbb{Z}+\alpha \;:=\; (hj+\alpha)_{j\in\mathbb{Z}}$.

Splines with a uniform knot sequence have been investigated much more extensively and for a much longer time than general polynomial splines. Not only was Schoenberg's fundamental paper [S] dedicated to such splines, but earlier mathematical literature from Quade and Collatz [QC], Kolmogorov [K], Eagle [E], back to Frobenius [F], Hermite and Sonin [HS] and even Laplace [L] has dealt with various aspects of splines with uniformly spaced knots. The thorough and detailed knowledge about such splines now available has been obtained largely through Fourier analysis, applicable because of their regular structure.

This regular structure allows one to simplify many of the results of the preceding chapter. We list the following examples.

Let $(N_{j,k})_j$ be the sequence of B–splines of order k for the knot sequence $\mathbb{Z}$. Then

$$(1.1)\quad N_{0,k}(t) = \sum_{\nu=0}^{k} (-)^{\nu} \binom{k}{\nu} (t-\nu)_+^{k-1}/(k-1)! \;= \sum_{\nu=0}^{k} (-)^{k-\nu} \binom{k}{\nu} (\nu-t)_+^{k-1}/(k-1)$$

and

$$(1.2)\qquad M_{j,k} = N_{j,k} = N_{0,k}(\cdot - j).$$

Differentiation of a B–spline series now simplifies to differencing, i.e., (I.4.2) becomes

$$(1.3)\qquad D^{\nu}\Big(\sum_j \alpha_j N_{j,k}\Big) = \sum_j \nabla^{\nu}\alpha_j N_{j,k-\nu}\ , \quad \nu = 0,\dots,k-1.$$

We will make extensive use of the second of Schoenberg's abbreviations

$$(1.4)\qquad Q_k \;:=\; N_{0,k}\ ,\quad M_k \;:=\; N_{0,k}(\cdot + k/2)\quad ,$$

who calls it the **forward**, respectively the **central**, B–spline of order k since

$$(1.5)\qquad \Delta^k f(0) = \int Q_k(s) f^{(k)}(s)\,\mathrm{d}s\ ,\quad \delta^k f(0) = \int M_k(s) f^{(k)}(s)\,\mathrm{d}s$$

gives the forward, respectively the central, difference of order k (with step 1). (Recall that $k!\,[0,\dots,k]f = \Delta^k f(0)$.)

In particular, the Fourier transform for M_k is easily seen to be

$$(1.6)\qquad \widehat{M_k}(t) \;:=\; \int_{-\infty}^{\infty} M_k(s)\,\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}s = \Big(\frac{\sin t/2}{t/2}\Big)^k$$

since, for $f(s) := e^{ist}$, $\delta^k f(s) = (2i\sin t/2)^k f(s)$ while $f^{(k)}(s) = (it)^k e^{ist}$. Hence, also

$$M_k = M_{k-r} * M_r := \int M_{k-r}(\cdot - s) M_r(s)\, ds \tag{1.7}$$

since the Fourier transform associates products with convolutions. It follows that M_k is the $(k-1)$–fold convolution of the characteristic function of $[-1/2\,..\,1/2]$ with itself. Consequently, M_k is the distribution of the sum of k independent random variables, each uniformly distributed on $[-1/2\,..\,1/2]$, for which reason these B–splines occur already in the works of Laplace.

We will use the abbreviation

$$\mathcal{S}_k := \Big\{ \sum_j \alpha_j M_k(\cdot - j) : \alpha_j \in \mathbb{C}\ , \quad \text{all } j \in \mathbb{Z} \Big\}. \tag{1.8}$$

With the choice $\tau_j = j + k/2$, all j, the identity

$$f = \sum_j \Big(\lambda_{\tau_j, \psi_{j,k}} f \Big) N_{j,k}\ , \quad \text{all } f \in \mathcal{S}_{k,\mathbb{Z}},$$

obtained from (I.2.2) Lemma and its corollary, translates into

$$f = \sum_j \lambda_k f(\cdot + j) M_k(\cdot - j)\ , \quad \text{all } f \in \mathcal{S}_k \tag{1.9a}$$

with

$$\lambda_k f = \sum_{\nu < k} (-)^\nu B_\nu^{(k)}(k/2)/\nu!\, f^{(\nu)}(0). \tag{1.9b}$$

Here, $B_\nu^{(k)}(t)$ denotes the Bernoulli polynomial of order k and degree ν, i.e., $(B_\nu^{(k)}/\nu! : \nu)$ is the Appell sequence for $k![0,\ldots,k]D^{-k}$, hence

$$e^{zt} \left(\frac{z}{e^z - 1} \right)^k = \sum_{\nu=0}^{\infty} B_\nu^{(k)}(t)/\nu!\, z^\nu \qquad (\text{ on } |z| < 2\pi) \tag{1.10}$$

is its generating function (see [N: p. 145]), and (1.9) follows from (I.2.2) Lemma via the fact [N: p. 148] that

$$B_\nu^{(n+1)}/\nu! = D^{n-\nu}(\cdot - 1)\cdots(\cdot - n)/n! \quad .$$

Note that (1.10) reduces to

$$\left(\frac{z/2}{\sin z/2} \right)^k = \sum_{\nu=0}^{\infty} B_\nu^{(k)}(k/2)/\nu!\, (iz)^\nu \tag{1.11}$$

for $t = k/2$, hence $B_\nu^{(k)}(k/2) = 0$ for odd ν. More specifically,

$$B_\nu^{(k)}(k/2) = D_\nu^{(k)}/2^\nu \tag{1.12}$$

(see [N: equ.(43) on p. 130]), with $D_\nu^{(k)}$ a polynomial of degree $\nu/2$ in k, the first few of which (see [N: p. 460]) are

(1.13)

ν	$D_\nu^{(k)}$
0	1
2	$-k/3$
4	$k(5k+2)/15$
6	$-k(35k^2+42k+16)/63$
8	$k(175k^3+420k^2+404k+144)/135$
10	$-k(385k^4+1540k^3+2684k^2+2288k+768)/99$

2. Cardinal spline interpolation

Cardinal spline interpolation is concerned with the construction and study of (partial) right inverses for the linear map

$$C_k : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}} : \boldsymbol{\alpha} \mapsto \Big(\sum_j \alpha_j M_k(\nu - j)\Big)_{\nu=-\infty}^{\infty} \tag{2.1}$$

and thereby for its factor

$$\mathcal{S}_k \to \mathbb{C}^{\mathbb{Z}} : \varphi \mapsto (\varphi(\nu))_{\nu=-\infty}^{\infty}.$$

Let

$$m := \lfloor (k-1)/2 \rfloor \quad .$$

Since $M_k(\nu) \neq 0$ iff $|\nu| < k/2$,

$$C_k \boldsymbol{\alpha} = \boldsymbol{\beta}$$

is a linear difference equation of order $2m$ with real constant coefficients, hence has solutions for arbitrary $\boldsymbol{\beta} \in \mathbb{C}^{\mathbb{Z}}$. These solutions can be constructed by choosing $\alpha_{-m}, \ldots, \alpha_{m-1}$ arbitrarily and then computing

$$\alpha_{\nu+m} = \Big(\beta_\nu - \sum_{j<\nu+m} \alpha_j M_k(\nu - j)\Big)/M_k(-m) \ , \quad \nu = 0, 1, \ldots$$
$$\alpha_{\nu-m} = \Big(\beta_\nu - \sum_{j>\nu-m} \alpha_j M_k(\nu - j)\Big)/M_k(m) \quad , \quad \nu = -1, -2, \ldots$$

In short,

2.2. *C_k is onto and has a kernel of dimension $2m$.*

On subspaces of sequences which do not grow too fast at infinity, an explicit inverse for C_k can be constructed as follows. $C_k\boldsymbol{\alpha}$ is given by the convolution of $\boldsymbol{\alpha}$ with the sequence $(M_k(\nu))_\nu$ which is trivially in ℓ_1; in fact, $\sum_\nu |M_k(\nu)| = \sum_\nu M_k(\nu) = 1$. Hence

$$\|C_k\boldsymbol{\alpha}\|_p \le \|\boldsymbol{\alpha}\|_p\ , \quad \text{for all } \boldsymbol{\alpha} \in \ell_p \quad .$$

Further, since

$$c_k(z) \ := \sum_{n=-\infty}^{\infty} M_k(n)\, z^n$$

does not vanish on $|z| = 1$ (see (2.4) below), there exists $\omega_k \in \ell_1$ (in fact, $\omega_k(j)$ goes to zero exponentially as $j \to \pm\infty$ by (2.9) below) such that

$$\sum_{j=-\infty}^{\infty} \omega_k(j) z^j = 1/c_k(z) \quad \text{on } |z| = 1 \quad .$$

But then

$$1 = \Big(\sum_\mu \omega_k(\mu) z^\mu\Big)\ \Big(\sum_\nu M_k(\nu) z^\nu\Big) = \sum_j \Big(\sum_{\mu+\nu=j} \omega_k(\mu) M_k(\nu)\Big) z^j \quad \text{on } |z| = 1$$

showing that

$$\sum_{\mu+\nu=j} \omega_k(\mu) M_k(\nu) = \delta_{0j}, \quad \text{all } j \in \mathbb{Z} \quad .$$

Hence,

2.3.

$$\ell_p \to \ell_p : \boldsymbol{\alpha} \mapsto \Big(\sum_j \alpha_j \omega_k(\nu - j)\Big)_{\nu=-\infty}^{\infty}$$

is the inverse for $C_k|_{\ell_p}$, and $\|(C_k|_{\ell_p})^{-1}\| \le \|\boldsymbol{\omega}_k\|_1, \quad 1 \le p \le \infty$.

Since M_k is real, even, and $M_k(t) = 0$ for $|t| \ge k/2$,

$$c_k(z) = \sum_{|j|<k/2} M_k(j) z^j$$

is a rational function with real coefficients, invariant under $z \mapsto 1/z$, hence real on $|z| = 1$.

2.4 Lemma. *With $m \ := \ \lfloor (k-1)/2) \rfloor$, the function $c_k(z) = \sum_j M_k(j) z^j$ has exactly $2m$ zeros, all negative and simple, with λ a zero iff $1/\lambda$ is.*

Proof: For $\tau \in (0\,..\,1]$, define the polynomial

$$\pi_{n,\tau}(z) \ := \sum_{j=0}^{n} N_{0,n+1}(n + 1 - \tau - j) z^j \quad .$$

Then

$$\pi_{n,\tau}(z) = \sum_j N_{0,n+1}(\tau+j)z^j = \sum_j M_{n+1}\big(\tau+j-(n+1)/2\big)z^j$$
$$= z^{(n+1)/2-\tau} \sum_j M_{n+1}\big(\tau+j-(n+1)/2\big)z^{\tau+j-(n+1)/2} \quad .$$

Hence

$$z^m c_k(z) = \begin{cases} \pi_{k-1,1/2}(z) & , \quad k \ \text{ odd} \\ \\ \pi_{k-1,1}(z) & , \quad k \ \text{ even} \end{cases} \quad ,$$

a selfreciprocal polynomial of degree $2m$. It is therefore sufficient to prove that, for $\tau \in (0\,..\,1]$ and all $n \geq 1$, all the zeros of $\pi_{n,\tau}$ are negative and simple. For this, we follow Schoenberg and introduce, for $z \in \mathbb{C}\backslash\{0\}$, the **exponential spline of degree n to the base z**,

$$\Phi_{n,z} \ := \ \sum_j z^j N_{j,n+1} \quad , \tag{2.5}$$

which spans the subspace of those $s \in \mathcal{S}_{n+1,\mathbb{Z}}$ satisfying the functional equation

$$s(t+1) = zs(t), \quad \text{all } t \in \mathbb{R} \quad .$$

In terms of this spline,

$$\pi_{n,\tau}(z) = \sum_{j=0}^{n} N_{j,n+1}(n+1-\tau)z^j = \Phi_{n,z}(n+1-\tau) = z^n \Phi_{n,z}(1-\tau) \quad ,$$

hence

$$\pi^{(1)}_{n,\tau}(z) = (n/z)\pi_{n,\tau}(z) + z^n(\partial/\partial z)\Phi_{n,z}(1-\tau) \quad . \tag{2.6}$$

Further, by (I.4.3),

$$\Phi_{n+1,z}(t) = \sum_j \left((1+\frac{z-1}{n+1}t) \ + \ \frac{1-z}{n+1}j\right) z^{j-1}\, N_{j,n+1}(t)$$

which, combined with (2.6), gives the recurrence relation

$$\begin{aligned} \pi_{n+1,\tau}(z) &= \Big(z+\tau\frac{1-z}{n+1}\Big)\pi_{n,\tau}(z) + z\frac{1-z}{n+1}\,\pi^{(1)}_{n,\tau}(z), \quad n>0 \\ \pi_{0,\tau}(z) &= 1 \end{aligned} \tag{2.7}$$

It follows that $\pi_{n+1,\tau}(0) = (\tau/(n+1))\pi_{n,\tau}(0) > 0$, while, for every negative zero λ of $\pi_{n,\tau}$, $\pi_{n+1,\tau}(\lambda)\pi^{(1)}_{n,\tau}(\lambda) < 0$ and, finally, $\pi_{n+1,\tau}$ has a positive leading coefficient. Therefore, induction on n gives that, for $\tau \in (0\,..\,1)$, $\pi_{n,\tau}$ has the n distinct and negative zeros $\lambda_{n,n}(\tau) < \cdots < \lambda_{1,n}(\tau)$ with the zeros of $\pi_{n-1,\tau}$ interlacing those of $\pi_{n,\tau}$, i.e., $\lambda_{i+1,n}(\tau) < \lambda_{i,n-1}(\tau) < \lambda_{i,n}(\tau)$. These statements remain true for $\tau = 1$ except that now $\lambda_{n,n}(1^-) = -\infty$, i.e., $\pi_{n,1}$ is only of degree $n-1$ and has, correspondingly, only $n-1$ zeros. □

2.8 Corollary. *For some* $\lfloor (k-1)/2 \rfloor$*–point subset* Γ_k *of* $(0\,..\,1)$,

$$c_k(z) = \prod_{\gamma\in\Gamma_k} \frac{1+\gamma z}{1+\gamma}\,\frac{1+\gamma/z}{1+\gamma} \quad .$$

It follows that the function

$$\varphi_k(\theta) \ := \ c_k(\mathrm{e}^{\mathrm{i}\theta})$$

is of the form

$$\varphi_k(\theta) = \mathrm{const}_k \prod_{\gamma\in\Gamma_k} f_\gamma(\theta)$$

where, for $\gamma \in (0\,..\,1)$,

$$f_\gamma(\theta) \ := \ (1+\gamma\mathrm{e}^{\mathrm{i}\theta})(1+\gamma\mathrm{e}^{-\mathrm{i}\theta}) = |1+\gamma\mathrm{e}^{-\mathrm{i}\theta}|^2$$

is positive, 2π–periodic and even, and strictly monotone decreasing from its maximum value at $\theta=0$ to its minimum value at $\theta=\pi$. Hence,

2.9. φ_k *is positive,* 2π*–periodic and even and, for* $k>2$*, strictly decreases from its maximum value* $1=\varphi_k(0)$ *to its positive minimum value* $\prod_{\gamma\in\Gamma_k}(1-\gamma)^2/(1+\gamma)^2 = \varphi_k(\pi)$*.*

Also, (2.8) implies that, near $|z|=1$,

$$\sum_n \omega_k(n) z^n = 1/c_k(z) = \prod_{\gamma\in\Gamma_k} \frac{1+\gamma}{1+\gamma z}\,\frac{1+\gamma}{1+\gamma/z}$$

hence

(2.10)
$$\left.\begin{array}{ll} \omega_k(n) = \omega_k(-n) \in \mathbb{R} & \\ \omega_k(n) = \delta_{0n} \quad , & k=1,2 \\ 0 < (-)^n\omega_k(n) < \mathrm{const}\big(\max_{\gamma\in\Gamma_k}\gamma\big)^{|n|}, & k>2 \end{array}\right\} \quad , \quad \text{all } n\in\mathbb{Z} \quad .$$

In particular,

(2.11)
$$\|\boldsymbol{\omega}_k\|_1 = 1/c_k(-1) = 1/\varphi_k(\pi) \quad ,$$

which, with (2.3), implies that

$$\|(C_k\big|_{\ell_p})^{-1}\| \le 1/\varphi_k(\pi) \quad .$$

Equality occurs here for $p=\infty$, the norm being taken on at the sequence $((-)^n)_{n\in\mathbb{Z}}$. Equality also occurs for $p=2$ since the unitary map

$$\ell_2 \to \mathbb{L}_2(-\pi,\pi] : \boldsymbol{\alpha} \mapsto \sum_n \alpha_n \mathrm{e}^{\mathrm{i}n\theta}/\sqrt{2\pi}$$

carries $C_k|_{\ell_2}$ to multiplication by φ_k. But neither $C_k|_{\ell_2}$ nor its inverse take on their norm since, by (2.9), φ_k fails to be constant on a set of positive measure, hence $C_k|_{\ell_2}$ has no point spectrum. By interpolation, $\|(C_k|_{\ell_p})^{-1}\|$ is a convex function of p on $1 \le p \le \infty$, therefore, as it equals its upper bound $1/\varphi_k(\pi)$ at $p = 2$ and $p = \infty$, it follows that

$$\|(C_k|_{\ell_p})^{-1}\| = 1/\varphi_k(\pi) \quad \text{for } 1 \le p \le \infty \quad . \tag{2.12}$$

The same argument also shows that

$$\|C_k|_{\ell_p}\| = \varphi_k(0) = 1, \quad 1 \le p \le \infty \quad . \tag{2.13}$$

The number

$$1/\varphi_k(\pi) = \prod_{\gamma \in \Gamma_k} (1+\gamma)^2/(1-\gamma)^2$$

is related to Favard's constant (see **???**) and admits various other representations. E.g., applying Poisson's summation formula

$$\sum_n f(x-n) = \sum_n \int_{-\infty}^{\infty} f(s)\, \mathrm{e}^{-2\pi \mathrm{i} sn}\, \mathrm{d}s\; \mathrm{e}^{2\pi \mathrm{i} xn}$$

at $x = 0$ to $f(s) := M_k(s)\,\mathrm{e}^{\mathrm{i}s\theta}$, one finds that

$$\varphi_k(\theta) = \sum_n M_k(n)\,\mathrm{e}^{\mathrm{i}n\theta} = \sum_n \widehat{M}_k(\theta + 2\pi n) \tag{2.14}$$

where, by (1.6), $\widehat{M}_k(u) = ((\sin u/2)/(u/2))^k$. Hence

$$1/\varphi_k(\pi) = \frac{1}{2}\left(\frac{\pi}{2}\right)^k \Bigg/ \sum_{n=0}^{\infty} (-)^{nk}/\,(2n+1)^k \quad . \tag{2.15}$$

Equivalently (see also),

$$\varphi_k(\pi) = \begin{cases} (-)^{k/2} B_k\, 2^k(1-2^k)/k! & , k \text{ even} \\ (-)^{(k-1)/2} E_{k-1}/(k-1)! & , k \text{ odd} \end{cases} \tag{2.16}$$

with B_j and E_j the Bernoulli and Euler numbers, respectively. The first few values are

(2.17)

k	$1/\varphi_k(\pi)$
1	1
2	1
3	2

k	$1/\varphi_k(\pi)$
4	3
5	4.8
6	7.5

k	$1/\varphi_k(\pi)$
7	720/61=11.803..
8	315/17=18.529..
9	8064/277=29.111..

and, already for $k = 10$, $1/\varphi_k(\pi)$ is within .002% of $(\pi/2)^k/2$.

By (2.8), $\ker C_k$ has a basis consisting of the $2m$ sequences

$$(\lambda^\nu)_{\nu\in\mathbb{Z}}, \quad \text{with } -\lambda \text{ or } -1/\lambda \quad \text{in } \Gamma_k \quad .$$

As $\Gamma_k \subseteq (0\,..\,1)$, it follows that $\boldsymbol{\alpha} \in \ker\, C_k\backslash\{0\}$ *implies that* $\gamma_k^n \max\{|\alpha_n|, |\alpha_{-n}|\}$ *fails to go to zero as* $n \to \infty$, with

$$\gamma_k \;:=\; \max_{\gamma\in\Gamma_k} \gamma \quad .$$

C_k is therefore 1-1 on any subspace of $\mathbb{C}^{\mathbb{Z}}$ whose elements do not grow too fast at $\pm\infty$.

To give an example, C_k is 1-1 on

$$\ell_{\infty,\gamma} \;:=\; \left\{\boldsymbol{\alpha} \in \mathbb{C}^{\mathbb{Z}} : \|\boldsymbol{\alpha}\|_{\infty,\gamma} \;:=\; \sup_\nu \gamma^{|\nu|}\,|\alpha_\nu| < \infty\right\}$$

for any $\gamma > \gamma_k$. Actually, much more is true. Define the **shift** E on $\mathbb{C}^{\mathbb{Z}}$ by the rule

$$(E\boldsymbol{\alpha})_\nu = \alpha_{\nu+1}, \quad \text{all } \nu \in \mathbb{Z} \quad .$$

Then E and E^{-1} map $\ell_{\infty,\gamma}$ into itself, hence so does E^n for all $n \in \mathbb{Z}$, while

$$\|E^n|_{\ell_{\infty,\gamma}}\| = \sup_{\boldsymbol{\alpha}} \frac{\sup_\nu \gamma^{|\nu|}\,|\alpha_{n+\nu}|}{\sup_\nu \gamma^{|\nu|}\,|\alpha_\nu|} \le \gamma^{-|n|}\,, \quad \text{all } n \in \mathbb{Z}$$

if $\gamma \in (0\,..\,1]$. Therefore, as

$$\|\boldsymbol{\omega}_k * \boldsymbol{\alpha}\|_{\infty,\gamma} = \|\sum_n \omega_k(-n)\,(E^n\boldsymbol{\alpha})\|_{\infty,\gamma} \le \sum_n |\omega_k(-n)|\,\|E^n\|\,\|\boldsymbol{\alpha}\|_{\infty,\gamma} \quad ,$$

one gets from (2.10) that

$$\|\boldsymbol{\omega}_k * \cdot\Big|_{\ell_{\infty,\gamma}}\| \le \sum_n |\omega_k(-n)|\gamma^{-|n|} \le \text{const} \sum_n (\gamma_k/\gamma)^n < \infty$$

provided $\gamma \in (\gamma_k\,..\,1]$. This proves a particular instance of the

2.18 Theorem. *If the linear subspace X of $\mathbb{C}^{\mathbb{Z}}$ is invariant under the shift E and its inverse E^{-1}, and if, with respect to some norm on X, the sequence $(\|E^n\|)_{n\in\mathbb{Z}}$ of corresponding norms satisfies*

$$\sum_n \|E^n\|\gamma_k^{|n|} < \infty \quad ,$$

then $C_k|_X$ is a bounded map from X onto X and has convolution with $\boldsymbol{\omega}_k$ as its inverse, with $\|(C_k|_X)^{-1}\| \le \sum_n \|E^n\|\gamma_k^{|n|}$.

A rather different description for $(C_k|_X)^{-1}$ can be given in case $X = \mathbb{P}_r|_{\mathbb{Z}}$. For $p \in \mathbb{P}_r$,

$$\sum_\mu p(\mu)\,M_k(\nu-\mu) = \sum_\mu p(\mu+\nu)\,M_k(-\mu) = \sum_j p^{(j)}\,(\nu) \sum_\mu \mu^j/j!\,M_k(-\mu)$$
$$= \sum_j p^{(2j)}(\nu)\,(-)^j\,\delta_{2j}^{(k)}$$

hence

$$C_k(p|_{\mathbb{Z}}) = \big(\sum_j (-)^j \delta_{2j}^{(k)}\, p^{(2j)}\big)|_{\mathbb{Z}}$$

with $(\delta_j^{(k)})_{j=0}^{\infty}$ the coefficients in the Taylor series expansion for $\varphi_k(\theta) = \sum_\mu \mathrm{e}^{\mathrm{i}\mu\theta}\, M_k(\mu)$ around $\theta = 0$, i.e.,

$$\varphi_k(\theta) = \sum_{j=0}^{\infty} \delta_{2j}^{(k)}\, \theta^{2j}$$

since φ_k is an even function. In particular, $\delta_0^{(k)} = \varphi_k(0) = 1$, hence

2.19. *C_k maps $\mathbb{P}_r|_{\mathbb{Z}}$ onto itself and preserves leading coefficients.*

I.e.,

$$(1 - C_k)\,\mathbb{P}_r|_{\mathbb{Z}} \subseteq \mathbb{P}_{r-1}|_{\mathbb{Z}} \quad .$$

C_k restricted to $\mathbb{P}_r|_{\mathbb{Z}}$ is therefore invertible and, since $\mathbb{P}_r|_{\mathbb{Z}} \subseteq \ell_{\infty,\gamma}$ for every $\gamma \in (\gamma_k \mathinner{..} 1)$ while

$$1/\varphi_k(u) = \sum_\mu \mathrm{e}^{\mathrm{i}\mu u} \omega_k(\mu) \quad ,$$

its inverse is given by

$$p|_{\mathbb{Z}} \mapsto \big(\sum_{j=0}^{\infty} (-)^j \gamma_{2j}^{(k)}\, p^{(2j)}\big)|_{\mathbb{Z}} \tag{2.20}$$

with

$$1/\varphi_k(\theta) = \sum_{j=0}^{\infty} \gamma_{2j}^{(k)}\, \theta^{2j} \quad . \tag{2.21}$$

Incidentally, (2.8) implies that $1/\varphi_k(u)$ is 2π–periodic with all its singularities in the period strip $0 \le \mathrm{Re}\ u < 2\pi$ on the line $\mathrm{Re}\ u = \pi$ and off the real line. The series (2.21) has therefore radius of convergence $\pi + B_k$ for some positive constant B_k which implies that

$$|\gamma_{2j}^{(k)}| \le A_k(\pi + B_k)^{-2j}, \quad \text{all } j \quad ,$$

for an appropriate constant A_k. Therefore:

2.22. *The map (2.20) is defined for every entire function p of exponential type $< \pi + B_k$ and produces a sequence $\boldsymbol{\alpha} = \boldsymbol{\alpha}_p$ so that*

$$C_k \boldsymbol{\alpha}_p = p|_{\mathbb{Z}}.$$

Finally, the map

$$f \mapsto \Big(\sum_{j<k} (-)^j\, B_j^{(k)}(k/2)/j!\, f^{(j)}\Big)|_{\mathbb{Z}}$$

carries each $f \in \mathcal{S}_k$ to the sequence of its B-spline coefficients, by (1.9), hence must agree with (2.20) whenever $f = p \in \mathbb{P}_k \subseteq \mathcal{S}_k$. This implies that

$$(-)^j \gamma_{2j}^{(k)} = B_{2j}^{(k)}(k/2)/(2j)!\,, \quad \text{all } 2j < k \quad ,$$

i.e., that

$$1/\varphi_k(\theta) = 1/\widehat{M_k}(\theta) + O(\theta^k) \quad . \tag{2.23}$$

3. The exponential splines

Cardinal spline interpolation can be extended to certain functions of faster than power growth with the aid of the exponential splines. For

$$z \in \mathbb{C}\backslash\{0,1\} \quad ,$$

the *exponential spline of degree* n *to the base* z was introduced in (2.5) as the function

$$\Phi_{n,z} \;:=\; \sum_j z^j N_{j,n+1}$$

which, up to scalar multiples, is the unique solution in $\mathcal{S}_{n+1,\mathbb{Z}}$ of the functional equation

$$f(t+1) = z f(t), \quad \text{all } t \in \mathbb{R} \quad .$$

If $\Phi_{n,z}(\tau) \neq 0$ for some $\tau \in [0\,..\,1)$, then

$$S_{n,z,\tau} \;:=\; z^\tau \Phi_{n,z}/\Phi_{n,z}(\tau) \tag{3.1}$$

is an element of $\mathcal{S}_{n+1,\mathbb{Z}}$ which agrees with the exponential function

$$z^t \;:=\; |z|^t\, \mathrm{e}^{\mathrm{i}t \arg\, z}, \quad t \in \mathbb{R} \quad ,$$

at the integer translates $\tau + k$, all $k \in \mathbb{Z}$, of τ.

The interpolant $S_{n,z,\tau}$ is defined for all bases $z \in \mathbb{C}\backslash\{0,1\}$ with $n-1$ or n exceptions. By (2.4) Lemma, the polynomial

$$\pi_{n,1-\tau}(z) = z^n \Phi_{n,z}(\tau)$$

in z has exactly $\lceil n-1+\tau \rceil$ zeros $z_\nu = z_\nu(n,\tau)$, all negative and simple. The corresponding exponential spline Φ_{n,z_ν} has been called an **eigenspline**, with z_ν its associated **eigenvalue**. Since z_ν is negative, such an eigenspline grows exponentially either for $t \to \infty$ or else for $t \to -\infty$ unless $z_\nu = -1$. Hence, τ is associated with a *bounded* eigenspline iff $z_\nu(n,\tau) = -1$ for some ν, i.e., iff $\Phi_{n,-1}(\tau) = 0$. This happens for odd n at $\tau = 1/2$ and nowhere else in $[0\,..\,1)$ and for even n at $\tau = 0$ and nowhere else in $[0\,..\,1)$ (see (3.5) below). In short, bounded eigensplines are very rare:

3.2. *If $\varphi \in \mathcal{S}_{n+1,\mathbb{Z}} \backslash \{0\}$ is of power growth and vanishes at $\tau + k$ for some $\tau \in [0\,..\,1)$ and all $k \in \mathbb{Z}$, then $\tau = 1/2$ and $\varphi = \alpha \mathcal{E}_n$ for odd n,* **and** *$\tau = 0$* **and** *$\varphi = \alpha \mathcal{E}_n(\cdot - 1/2)$ for even n.*

Here, $\mathcal{E}_n$ is the **Euler spline**,

$$\mathcal{E}_n \;:=\; \sum_j (-)^j M_{n+1}(\cdot - j)/\varphi_{n+1}(\pi); \tag{3.3}$$

i.e., the bounded cardinal spline interpolant to the sequence $((-)^\nu)_{\nu \in \mathbb{Z}}$, with $\varphi_{n+1}(\pi) = \sum_j (-)^j M_{n+1}(-j)$ described in detail in (2.15)–(2.17).

In order to prove the above mentioned fact about the zeros of $\Phi_{n,-1}$, and for a better understanding of exponential splines, consider now the polynomial $\Phi_{n,z}|_{(0..1)}$. Since $\nabla z^j = z^j(1 - 1/z)$, (1.3) implies that

$$\Phi_{n,z}^{(\nu)} = (1 - 1/z)^\nu \, \Phi_{n-\nu,z}, \quad \nu = 0, \ldots, n \quad .$$

Hence, with $A_{n,z}$ the polynomial of degree $\le n$ for which

$$A_{n,z} = \Phi_{n,z}/(1 - 1/z)^n \quad \text{on } (0\,..\,1), \tag{3.4}$$

we have

$$A_{n,z}^{(\nu)}(1) = z A_{n,z}^{(\nu)}(0), \quad \nu = 0, \ldots, n-1; \quad A_{n,z}^{(n)} = 1$$

showing that $(A_{n,z})_{n=0}^\infty$ is the Appell sequence for the linear functional

$$\mu_z \;:=\; ([1] - z[0])/(1 - z) \quad ,$$

i.e., $(A_{n,z})_n$ is the unique sequence of polynomials satisfying

$$\left.\begin{array}{l} \text{degree } A_{n,z} \le n \\ \mu_z D^m A_{n,z} = \delta_{m,n} \end{array}\right\} , \quad \text{all } m, n \ge 0 \quad .$$

This can be said slightly differently: $A_{n,z}$ is the unique polynomial solution (necessarily of degree n) for the difference equation

$$\frac{A_{n,z}(t+1) - z A_{n,z}(t)}{1 - z} = t^n/n! \quad .$$

In the special case $z = -1$, this becomes the well-known equation

$$\Big(A_{n,-1}(t+1) + A_{n,-1}(t)\Big)/2 = t^n/n!$$

whose solution is $A_{n,-1} = E_n/n!$, with E_n the Euler polynomial of degree n (see [N: p. 23ff]). Since $(A_{n,-1})$ is the Appell sequence for $\mu_{-1} = ([0] + [1])/2$, it follows that

$$A_{n,-1} = \int_{1/2} A_{n-1,-1}(s)\,\mathrm{d}s - \mu_{-1} \int_{1/2} A_{n-1,-1}(s)\,\mathrm{d}s.$$

Further, μ_{-1} annihilates functions odd around 1/2, while $\int_{1/2}$ maps functions odd {even} around 1/2 to functions even {odd} around 1/2. Hence, $E_n/n! = A_{n,-1}$ is odd or even around 1/2 as n is odd or even. In particular, $A_{n,-1}(1/2) = 0$ for odd n, while, for even n, $A_{n,-1}(0) = A_{n,-1}(1)$, but also $0 = \mu_{-1} A_{n,-1} = (A_{n,-1}(0) + A_{n,-1}(1))/2$, therefore $A_{n,-1}(0) = 0$. Finally, if $A_{n-1,-1}$ has no other zeros in $[0\,..\,1]$, – and this is certainly so for $n = 2, 3$, – then $A_{n,-1}$ can have no other zeros, by Rolle's Theorem. This proves:

3.5 Result. *For* $\genfrac{}{}{0pt}{}{\text{even}}{\text{odd}}$ n*,* $\Phi_{n,-1}$ *is* $\genfrac{}{}{0pt}{}{\text{even}}{\text{odd}}$ *around each half integer* $k+1/2$*, and* $\genfrac{}{}{0pt}{}{\text{odd}}{\text{even}}$ *around each integer* k*, and vanishes in* $[0\,..\,1)$ *only at* $\genfrac{}{}{0pt}{}{0}{1/2}$*. Correspondingly, the Euler spline* $\mathcal{E}_n$ *is* $\genfrac{}{}{0pt}{}{\text{even}}{\text{odd}}$ *around each* $\genfrac{}{}{0pt}{}{}{\text{half}}$ *integer, and vanishes only at the half integers.*

The Euler spline was given its name (by Schoenberg) because of the fact that it, like $\Phi_{n,-1}$, is composed of (properly scaled) Euler polynomials.

For $n=0$, $A_{n,z}=1$. For $n>0$, one obtains a convenient expansion for $A_{n,z}$ from the generating function $\mathrm{e}^{wt}/\mu_z(\mathrm{e}^{w\cdot})$ for the Appell sequence $(A_{n,z})$, i.e., from

$$\mathrm{e}^{wt}\frac{1-z}{\mathrm{e}^{w}-z}=\sum_{\nu=0}^{\infty}A_{\nu,z}(t)\,w^{\nu}\quad. \tag{3.6}$$

On dividing (3.6) by w^{n+1}, one finds that the function

$$F(w)\;:=\;\mathrm{e}^{wt}\,(1-z)/\big((\mathrm{e}^{w}-z)w^{n+1}\big)=\sum_{\nu=0}^{\infty}A_{\nu,z}(t)\,w^{\nu-n-1}$$

is meromorphic, with a residue of

$$A_{n,z}(t)$$

at its pole of order $n+1$ at 0, and, for $\nu\in\mathbb{Z}$, a residue of

$$\mathrm{e}^{w_\nu t}(1-z)/(zw_\nu^{n+1})$$

at its simple pole at

$$w_\nu\;:=\;\ln|z|+\mathrm{i}(\arg z+2\pi\nu)\quad,$$

and no other poles. Hence

$$A_{n,z}(t)+(z^{-1}-1)\sum_{\nu=-N}^{N}\mathrm{e}^{w_\nu t}/w^{n+1}=\oint_{S_N}F(w)\,\mathrm{d}w$$

with S_N the boundary of the square of side length $(4N+2)\pi$ centered at z. Since, for $t\in[0\,..\,1]$, the line integral goes to zero as $N\to\infty$, while $A_{n,z}=\Phi_{n,z}/(1-1/z)^n$, we conclude that

$$\Phi_{n,z}(t)=(1-1/z)^{n+1}\,z^t\sum_{\nu\in\mathbb{Z}}w_\nu^{-n-1}\,\mathrm{e}^{2\pi\mathrm{i}\nu t} \tag{3.7a}$$

where

$$w_\nu=\ln|z|+\mathrm{i}(\arg z+2\pi\nu)\quad. \tag{3.7b}$$

Note that (3.7a) is valid for all $t\in\mathbb{R}$ since we just proved it for $t\in[0\,..\,1]$ and both sides of (3.7a) satisfy the functional equation $f(t+1)=zf(t)$.

It follows that the interpolant (3.1) for z^t is of the form

$$S_{n,z,\tau}(t)=z^t\Omega_{n,z}(t)/\Omega_{n,z}(\tau) \tag{3.8a}$$

with $\Omega_{n,z}$ the 1-periodic function given in terms of its Fourier series as

$$\Omega_{n,z}(t)\;:=\;\sum_{\nu\in\mathbb{Z}}\mathrm{e}^{2\pi\mathrm{i}\nu t}/w_\nu^{n+1}\quad. \tag{3.8b}$$

4. Cardinal spline interpolation as the degree goes to infinity

5. Approximation of cardinal type

The interpolation process

$$(Wf)(x) \ := \ \sum_{n=\in \mathbb{Z}} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)}$$

was studied in great detail by de la Vallée-Poussin [LVP] who called it the **fundamental** interpolation formula. The process was later used by Whittaker [W] for which reason the series is now known as **Whittaker's cardinal series**. It served Schoenberg [S] as the prototype for **cardinal interpolation**,

$$f \sim \sum_{n\in \mathbb{Z}} f(n) \, L(\cdot - n)$$

with $L(m) = \delta_{0m}$. In this section, we consider, more generally, **approximators of cardinal type**, i.e., linear maps of the form

(5.1) $$T = T_{L,\lambda} \ := \ \sum_{n\in \mathbb{Z}} E^{-n} \, L \otimes \lambda E^n$$

with

$$|L(x)| \le A \, \mathrm{e}^{-B|x|} \quad \text{for some positive } A, B,$$
$$\lambda \in C^*(\mathbb{R})$$

and E now also denoting the **unit shift**,

(5.2) $$(Ef)(x) \ := \ f(x+1)$$

for functions f on $\mathbb{R}$.

For such a map T, let

(5.3) $$T^{(h)} \ := \ S_{1/h} T S_h$$

with S_α denoting the α–**stretch**,

(5.4) $$(S_\alpha f)(x) \ := \ f(\alpha x) \quad .$$

Then $T^{(h)} f$ provides an approximation to f whose convergence behavior as $h \to 0$ reflects the local behavior of f. The following lemma is typical.

5.5 Lemma. *If λ has bounded support and the map $T = T_{L,\lambda}$ of cardinal type reproduces $\mathbb{P}_k$, then*

$$|\big(f - T^{(h)} f\big)(x)| \le \mathrm{const}_T \, h^{k-1} \, \omega\big(f^{(k-1)}, h; x\big)$$

whenever $f : \mathbb{R} \to \mathbb{C}$ has $k-1$ continuous derivatives and satisfies

$$\lim_{|x|\to\infty} \left|\mathrm{e}^{-B|x|} \, f(x)\right| = 0 \quad .$$

Proof:

Cardinal monosplines of least $\mathbb{L}_\infty$–norm are the topic of Schoenberg and Ziegler [SZ]. Here, the problem is to minimize $\|\cdot\|_\infty$ over

$$\mathbb{M}_m^r := \Big\{ f \in \mathbb{P}_{m+1,\mathbb{Z}} \cap C^{(r)}(\mathbb{R}) : D^m f = m! \Big\} \quad ,$$

i.e., to find

$$\alpha_{m,r} := \inf \Big\{ \|f\|_\infty : f \in \mathbb{M}_m^r \Big\}$$

for $-1 \le r \le m-2$. The number $\alpha_{m,r}$ is finite since $\mathbb{M}_m^r$ contains the 1–periodic extension $\overline{B}_m$ to all of $\mathbb{R}$ of the m–th Bernoulli polynomial $B_m|_{[0..1]}$ on $[0\,..\,1]$, the **Bernoulli monospline**. This allows one to rewrite $\mathbb{M}_m^r$ as

$$\mathbb{M}_m^r = \overline{B}_m - \mathbb{P}_{m,\mathbb{Z}} \cap C^{(r)}(\mathbb{R}) \quad .$$

It follows that $\mathbb{M}_m^r$ is **shift-invariant** (i.e., invariant under the unit shift E) and closed under uniform convergence on compact sets. Hence

Lemma. *For all $f \in \mathbb{M}_m^r$ there exists $g \in \mathbb{M}_m^r$ so that $\|g\|_\infty \le \|f\|_\infty$ and $Eg = g$.*

Proof: The map $A_n := \sum_{j=0}^{n-1} E^j/n$ leaves $\mathbb{M}_m^r$ invariant and is norm reducing, therefore $(A_n f)_n$ has limit points (under uniform convergence on compact sets) and all such limit points g lie in $\mathbb{M}_m^r$ and have norm no bigger than $\|f\|_\infty$. On the other hand,

$$(E-1)\, A_n = \sum_{j=1}^{n} E^j/n - \sum_{j=0}^{n-1} E^j/n = (E^n - 1)/n \xrightarrow[n\to\infty]{} 0 \quad \text{in norm} \quad ,$$

hence all limit points g of $(A_n f)$ are also in $\ker(E-1)$. □

A simpler argument shows that, with R reflection across $t = 1/2$, i.e.,

$$(Rf)(t) = f(1-t) \quad ,$$

there exists, for every $f \in \mathbb{M}_m^r \cap \ker(E-1)$, a $g \in \mathbb{M}_m^r \cap \ker(E-1)$, viz.

$$g = (f + (-)^m Rf)/2 \quad ,$$

with $\|g\|_\infty \le \|f\|_\infty$ and $(-)^m Rg = g$. Consequently,

$$\begin{aligned} \inf \Big\{ \|f\|_\infty : f \in \mathbb{M}_m^r \Big\} &= \inf \Big\{ \|g\|_\infty : g \in \mathbb{M}_m^r \quad \text{and} \quad Eg = g \quad \text{and} \quad (-)^m Rg = g \Big\} \\ &= \inf_{q \in Q_m^r} \|B_m - q\|_{\infty,[0..1/2]} \end{aligned}$$

with

$$Q_m^r := \Big\{ q \in \mathbb{P}_m : (-)^m Rq = q \; ; \quad D^j q(0) = D^j q(1) \, , \quad j = 0, \ldots, r \Big\} \quad .$$

The finite dimensionality of Q^r_m ensures that the infimum is attained.

In order to construct an element of $\mathbb{M}^r_m$ of minimal norm, it is convenient and natural to use the Bernoulli polynomials (B_n) since (see Appendix) $(B_n)_0^{m-1}$ provides a basis for $\mathbb{P}_m$ with the following properties:

$$B_0 = 1 \; ; \quad DB_n = nB_{n-1} \; ; \quad RB_n = (-)^n B_n \; ; \quad B_n(0) = B_n(1) \quad \text{for} \quad n \neq 1 \quad ,$$

hence

$$D^j B_n(0) = D^j B_n(1) \quad \text{for} \quad n - j \neq 1 \quad .$$

It follows that

$$Q^r_m = \text{span}\left\{ B_j : j = 0 \quad \text{or} \quad r+1 < j < m \; ; \quad j \equiv m (\text{mod} 2) \right\} \quad .$$

Explicitly, with

$$n \; := \; \lfloor (m-r)/2 \rfloor \quad ,$$

every $q \in Q^r_m$ is of the form

$$q = \sum_{j=1}^{n-1} a_{m-2j} B_{m-2j} + a_0$$

with $a_0 = 0$ in case m is odd.

Lemma. *Let $n \; := \; \lfloor (m-r)/2 \rfloor \geq 2$. For even m, every $q \in Q^r_m \backslash \{0\}$ is even around $1/2$ and has $\leq n-1$ zeros in $[0\,..\,1/2]$, hence $\leq 2(n-1)$ zeros in $[0\,..\,1]$. For odd m, every $q \in Q^r_m \backslash \{0\}$ is odd around $1/2$, vanishes at 0 and at $1/2$ and has $\leq n-2$ zeros in $(0, 1/2)$, hence $\leq 2n-1$ zeros in $[0\,..\,1]$.*

Proof: by induction on $m - 2n$. Let m be even, $q \in Q^r_m \backslash \{0\}$ with p zeros in $[0\,..\,1/2]$. Then q has $2p$ zeros in $[0\,..\,1]$. If $m - 2n = 0$, then q has degree $\leq m-2 = 2n-2$, hence $p \leq n-1$. Otherwise, $m - 2n \geq 2$ and $Dq \in Q^{r-1}_{m-1} \backslash \{0\}$, – we assume that $p > 0$, – and Dq has $2p-1$ zeros in $[0\,..\,1]$ in addition to its zeros at 0 and at 1, therefore $p \leq n-1$, by induction hypothesis.

If, on the other hand, m is odd and $q \in Q^r_m \backslash \{0\}$ has p zeros in $(0, 1/2)$, then q has $2p+3$ zeros in $[0\,..\,1]$. If $m - 2n = 1$, then q is of degree $\leq m-2 = 2n-1$ and therefore $p \leq n-2$. Otherwise, $m - 2n \geq 3$, hence $Dq \in Q^{r-1}_{m-1} \backslash \{0\}$ and has at least $2p+2$ zeros in $[0\,..\,1]$, hence $p \leq n-2$ by induction hypothesis. □

Theorem. *If m is even, then Q^r_m is a Chebyshev set on $[0\,..\,1/2]$, hence $B_m - Q^r_m$ contains a unique element $B_{m,r}$ of smallest uniform norm on $[0\,..\,1/2]$, characterized by the fact that for some $\varepsilon \in \{-1, 1\}$ and for some $0 = t_1 < \cdots < t_{n+1} = 1/2$,*

$$\varepsilon B_{m,r}(t_i) = (-)^i \|B_{m,r}\|_{\infty,[0..1/2]} \quad .$$

Proof: Since $\dim Q^r_m = n \; := \; \lfloor (m-r)/2 \rfloor$, the lemma ensures that Q^r_m satisfies Haar's condition on $[0\,..\,1/2]$. Standard arguments (see) therefore give existence and uniqueness of a best approximation to B_m in Q^r_m, characterized by alternation of its error $B_{m,r}$ on some $n+1$ points in $[0\,..\,1/2]$. But since $DB_{m,r}$ is in $Q^{r-1}_{m+1} \backslash \{0\}$ and vanishes at each of these points, only $n-1$ of these can lie in $(0\,..\,1/2)$, by Lemma above. □

Note that $Q_m^{m-2n-1} = Q_m^{m-2n}$, hence $B_{m,m-2n-1} = B_{m,m-2n}$. But, while the 1–periodic extension $\overline{B}_{m,m-2n}$ of $B_{m,m-2n}$ to all of $\mathbb{R}$ provides the **unique** element of smallest uniform norm in $\mathbb{M}_m^{m-2n}$, no such uniqueness is known for $\overline{B}_{m,m-2n-1} = \overline{B}_{m,m-2n}$ as an element of $\mathbb{M}_{m,m-2n-1}$.

Corollary. *With $\overline{B}_{m,m-2n}$ the 1–periodic extension of $B_{m,m-2n}|_{[0..1]}$ to all of $\mathbb{R}$, $f^* := \overline{B}_{m,m-2n}$ is the unique element of smallest uniform norm in $\mathbb{M}_m^{m-2n}$.*

Proof: By the theorem, there is an $\varepsilon \in \{-1, 1\}$ so that

$$\varepsilon f^*(t_i) = (-)^i \|f^*\|_\infty, \quad \text{all} \quad i \in \mathbb{Z} \quad ,$$

with $0 = t_1 < \cdots < t_{n+1} = 1/2$, and $t_{n+1+i} = 1 - t_{n+1-i}$, $i = 1, \ldots, n-1$, and, finally, $t_{i+2nk} = t_i + k$, all $i, k \in \mathbb{Z}$. Hence, if $f \in \mathbb{M}_m^{m-2n}$ and $\|f\|_\infty \le \|f^*\|_\infty$, then

$$s := f^* - f$$

is an element of $\mathbb{P}_{m,\mathbb{Z}} \cap C^{(m-2n)}(\mathbb{R})$ satisfying

$$\varepsilon s(t_i)\,(-)^i \ge 0 \quad \text{for all } i \in \mathbb{Z} \quad .$$

If $m - 2n = 0$, then, on each $[k \,..\, k+1]$, s is a polynomial of degree $< m$ with $m+1$ weak sign changes, therefore identically zero; hence $s = 0$ and so $f = f^*$.

If $m - 2n > 0$, then s is in $C^{(1)}$, therefore

$$s(t_i) = 0 \Longleftrightarrow t_i \quad \text{is (at least) a double zero of } s$$

as then t_i is an extremum for both f and f^*. Hence, if we allot one such zero to the interval $(t_{i-1} \,..\, t_i)$ and the "other" zero to the interval $(t_i \,..\, t_{i+1})$ in case $s(t_i) = 0$, then it follows that s has at least j zeros in $[t_i \,..\, t_{i+j}]$. In particular, s has $2nk$ zeros in $[0 \,..\, k]$ counting multiplicities. If now $s \ne 0$, then all of these zeros must be isolated. For, otherwise, s would vanish identically on some interval $[i \,..\, i+1]$ yet not vanish identically on $[i-1 \,..\, i]$, hence the polynomial $s|_{[i-1..i]}$ would be of nonnegative degree $< m$, yet would vanish $2n$ times on $[i-1 \,..\, i]$ in addition to its $(m-2n+1)$–fold zero at i (due to the fact that $s \in C^{(m-2n)}$), making it vanish m times, an impossibility. It follows that s is a spline of order m on $[0 \,..\, k]$ with $2nk$ isolated zeros (counting multiplicities), yet has only $k-1$ knots, each of multiplicity $m - (m-2n+1) = 2n-1$, an impossibility for large k by I.7.( ). This proves $s = 0$ and so $f = f^*$. □

The development for odd m proceeds along similar lines. We merely state the results.

Theorem. *If m is odd and $n := \lfloor (m-r)/2 \rfloor$, then $Q_m^r = \{vw : w \in Q\}$, with $v(t) := t(t-1/2)$ and Q a linear space of polynomials of dimension $n-1$ which satisfies Haar's condition on $(0 \,..\, 1/2)$. Since also B_m vanishes at 0 and at $1/2$, $B_m - Q_m^r$ therefore contains exactly one element $B_{m,r}$ of smallest uniform norm on $[0 \,..\, 1/2]$, and this element is characterized by the fact that, for some $\varepsilon \in \{-1, 1\}$ and some $0 < t_1 < \cdots < t_n < 1/2$,*

$$\varepsilon B_{m,r}(t_i) = (-)^i \|B_{m,r}\|_{\infty,[0..1/2]} \quad .$$

Further, $B_{m,m-2n-1} = B_{m,m-2n}$*, and the* 1*–periodic extension* $\overline{B}_{m,m-2n}$ *of* $B_{m,m-2n}|_{[0..1]}$ *to all of* $\mathbb{R}$ *is the unique element of minimal uniform norm in* $\mathbb{M}_m^{m-2n}$.

Schoenberg and Ziegler [SZ] call $B_{m,r}$ a BT or **Bernoulli–Chebyshev** polynomial, since $B_{m,-1}$ is given by

$$B_{m,-1}(t) = 2^{-2m+1}T_m(2t-1)$$

with T_m the m–th degree Chebyshev polynomial, while, on the other end of the scale,

$$B_{m,m-2} = B_m - B_m(0)/2^m \quad .$$

The BT polynomial $B_{m,m-2n} = B_{m,m-2n-1}$ is the unique element of minimal max norm on $[0\,..\,1]$ in

$$\Big\{p \in \mathbb{P}_{m+1} : D^m p = m!\,; \quad D^j p(0) = D^j p(1), \quad j = 0, \ldots, m-2n\Big\} \quad .$$

Qualitative information about $B_{m,r}$ can be obtained as follows: If m is even and $r = m-2n$, then, by Theorem, $B_{m,r}$ changes sign strongly on $0 = t_1 < \cdots < t_{2n+1} = 1$, i.e., $B_{m,r}(t_i)\, B_{m,r}(t_{i+1}) < 0$ for $i = 1, \ldots, 2$. Hence, there exist $0 < t_1^{(1)} < \cdots < t_{2n}^{(1)} < 1$ on which $DB_{m,r}$ changes sign strongly. Also, if $r \ge 2$, then $DB_{m,r}$ vanishes at 0 and at 1, therefore there exist points $0 \le t_1^{(2)} < \cdots < t_{2n+1}^{(2)} \le 1$ on which $D^2 B_m$ changes sign strongly. In fact we can take $t_1^{(2)} = 0$ and $t_{2n+1}^{(2)} = 1$ since otherwise $D^2 B_{m,r}$ would have to have more zeros in $[0\,..\,1]$ than are allowed by Lemma. In particular, as $|B_{m,r}|$ must have an extremum at $t = 0$, we must have

$$B_{m,r}(0)\, D^2 B_{m,r}(0) < 0 \quad .$$

Proceeding in this way, we find that, for $s = 1, \ldots, r/2$,

$$D^{2s-1} B_{m,r}(0) = 0, \quad D^{2s-2} B_{m,r}(0)\, D^{2s} B_{m,r}(0) < 0$$

and that $D^{2s} B_{m,r}$ changes sign strongly on some $0 = t_1^{(2s)} < \cdots < t_{2n+1}^{(2s)} = 1$. In particular, as $D^r B_{m,r}$ is of degree $m - r = 2n$, it has $2n$ real zeros, all simple and in $(0\,..\,1)$ and symmetric around $1/2$ since $D^r B_{m,r}$ is even around $1/2$. The Budan–Fourier Theorem (see I.7.) then implies that, because $D^{2n} D^r B_{m,r} = D^m B_{m,r} = m! > 0$, we must have

$$(-)^j D^{r+j} B_{m,r}(0) > 0, \quad D^{r+j} B_{m,r}(1) > 0, \quad j = 0, \ldots, 2n,$$

therefore also

$$(-) D^{r-2j} B_{m,r}(0) > 0, \quad j = 0, \ldots, r/2$$

and, in particular, $(-)^{r/2} B_{m,r}(0) > 0$, confirming the conjecture (2.18) of Schoenberg and Ziegler [SZ].

Analogous considerations show that, for odd m and $r = m-2n$, $D^r B_{m,r}$ changes sign strongly on some $0 = t_1^{(r)} < \cdots < t_{2n+1}^{(r)} = 1$, hence has $2n$ zeros in $[0\,..\,1]$, necessarily all simple since $D^r B_{m,r}$ has degree $2n$. Hence, again

$$(-)^j D^{r+j} B_{m,r}(0) > 0, \quad D^{r+j} B_{m,r}(1) > 0, \quad j = 0, \ldots, 2n \quad ,$$

and, finally

$$(-)^j D^{r-2j} B_{m,r}(0) > 0, \quad D^{r-2j-1} B_{m,r}(0) = 0, \quad j = 0, \ldots, (r-1)/2$$

and so, with t_1 the leftmost extremum of $B_{m,r}$ in $[0\,..\,1]$, we have $(-)^{(r-1)/2} B_{m,r}(t_1) > 0$, confirming conjecture (2.18) of Schoenberg and Ziegler [SZ].

Cardinal perfect splines of least $\mathbb{L}_\infty$–norm are treated in Cavaretta [C] in close imitation of Schoenberg and Ziegler's [SZ] treatment of the cardinal monosplines of least $\mathbb{L}_\infty$–norm just discussed. The problem now is to minimize $\|\cdot\|_\infty$ over

$$\mathbb{P}^r_m := \Big\{ f \in \mathbb{P}_{m+1,\mathbb{Z}} \cap C^{(r)}(\mathbb{R}) : D^m f = (-)^k m! \quad \text{on} \quad (k\,..\,k+1), \quad \text{all} \quad k$$

for $-1 \le r \le m-1$. This set is invariant under $-E$ rather than the unit shift E, hence it is sufficient to minimize $\|\cdot\|_\infty$ over

$$\mathbb{P}^r_m \cap \ker(E+1) \cap \ker\big((-)^m R - 1\big) \quad ,$$

– here R is again reflection across $t = 1/2$. The role played earlier by the Bernoulli polynomials is now taken over by the Euler polynomial (E_n), since $(E_n)_0^{m-1}$ provides a basis for $\mathbb{P}_m$ which satisfies (see Appendix)

$$E_0 = 1; \quad DE_n = E_{n-1}; \quad RE_n = (-)^n E_n; \quad E_n(0) = -E_n(1) \quad \text{for} \quad n \ne 0 \quad ,$$

hence

$$D^j E_n(0) = -D^j E_n(1) \quad \text{for} \quad n - j \ne 0 \quad .$$

For given f defined (at least) on $[0\,..\,1)$, let $\overline{f}$ denote the function defined on all of $\mathbb{R}$ which satisfies

$$\begin{aligned} \overline{f}(t) \quad &= \quad f(t) \qquad \text{for} \quad 0 \le t < 1 \\ \overline{f}(t+1) &= -\overline{f}(t) \quad \text{for all} \quad t \in \mathbb{R} \quad . \end{aligned}$$

In particular, $\overline{E}_m$ is an **Euler spline** normalized to have "leading" coefficient 1. Then the facts are as follows:

We have

$$\inf_{f \in \mathbb{P}^r_m} \|f\|_\infty = \min_{q \in Q^r_m} \|\overline{E}_m - q\|_{\infty,[0..1/2]}$$

with

$$\begin{aligned} Q^r_m \;:&= \Big\{ q \in \mathbb{P}_m : (-)^m Rq = q; \quad D^j q(0) = -D^j q(1) \quad \text{for} \quad j = 0, \ldots, r \Big\} \\ &= \operatorname{span}\Big\{ E_j : r < j < m; \quad j \equiv m \pmod 2 \Big\} \quad . \end{aligned}$$

Let

$$n := \lfloor (m - r - 1)/2 \rfloor \quad .$$

For even m, $Q^r_m = \{vw : w \in Q\}$, with $v(t) = t$ and Q a linear space of polynomials of dimension n satisfying Haar's condition on $(0\,..\,1/2]$. Hence, $E_m - Q^r_m$ contains exactly one element $E_{m,r}$ of smallest uniform norm on $[0\,..\,1/2]$, and this element is characterized by the fact that, for some $\varepsilon \in \{-1, 1\}$ and some $0 < t_1 < \cdots < t_{n+1} = 1/2$,

$$\varepsilon E_{m,r}(t_i) = (-)^i \|E_{m,r}\|_{\infty,[0..1/2]} \quad .$$

For odd m, $Q_m^r = \{vw : w \in Q\}$, with $v(t) = t - 1/2$ and Q a linear space of polynomials of dimension n satisfying Haar's condition on $[0\,..\,1/2)$. Hence, $E_m - Q_m^r$ contains exactly one element $E_{m,r}$ of smallest uniform norm on $[0\,..\,1/2]$, and this element is characterized by the fact that, for some $\varepsilon \in \{-1, 1\}$ and some $0 = t_1 < \cdots < t_{n+1} < 1/2$,

$$\varepsilon E_{m,r}(t_i) = (-)^i \|E_{m,r}\|_{\infty,[0..1/2]} \quad .$$

Cavaretta [loc.cit] calls $E_{m,r}$ an **Euler-Chebyshev**, or ET, polynomial since

$$E_{m,-1}(t) = 2^{-2m+1} T_m(2t-1) \quad \text{and} \quad E_{m,m-1} = E_m \quad .$$

Further, for $r = m - 2n - 1$, $E_{m,r} = E_{m,r-1}$, and $\overline{E}_{m,r}$ is the unique element of smallest uniform norm in $\mathbb{P}_m^r$. Also, $E_{m,r}$ is the unique element of smallest uniform norm in

$$\Big\{ p \in \mathbb{P}_{m+1} : D^j p(0) = -D^j p(1), \quad j = 0, \ldots, r \Big\} \quad .$$

The qualitative behaviour of $E_{m,r}$ could easily be elaborated further.

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