The algebra of products of linear polynomials

The fact that directional derivatives (rather than specific partial derivatives) are the handy tools in the analysis of functions in several variables is reflected in the fact that products of arbitrary linear homogeneous factors rather than just the powers x^α are preferred in work with multivariate polynomials.

We work with polynomials

$$p_Y : \mathbb{F}^d \to \mathbb{F} : x \mapsto \prod_{y\in Y} y\cdot x,$$

with Y a multiset or sequence in $\mathbb{F}$. For any $a \in \mathbb{F}^d$,

$$D_a p_Y = \sum_{y\in Y} y\cdot a\, p_{Y\backslash y},$$

hence

$$D_b D_a p_Y = \sum_{y\in Y} y\cdot a \sum_{z\in Y\backslash y} z\cdot b\, p_{Y\backslash\{y,z\}}.$$

Hence, for $\#A \le \#Y$,

$$D_A p_Y = \sum_{Z\in\binom{Y}{\#A}} p_{Y\backslash Z} \sum_{f\in \mathbf{S}_{\#A}} \prod_i z_i\cdot a_{f(i)},$$

while, for $\#A > \#Y$, $D_A p_Y = 0$.

In particular,

$$(D_a)^{\#Y} p_Y = (\#Y)!\, p_Y(a).$$

Fact. *The space Π^0_n of homogeneous polynomials of degree n is spanned by $(p_a)^n$, $a \in \mathbb{F}^d$.*

Proof: Observe that

$$(p_a)^n = \sum_{|\beta|=n} \binom{n}{\beta} a^\beta ()^\beta,$$

hence any linear functional λ_α, applied to $(p_a)^n$ as a function of a, with the property that, for $|\beta| = n$, $\lambda_\alpha()^\beta \ne 0$ iff $\beta = \alpha$ will produce a nontrivial multiple of $()^\alpha$, thus placing $()^\alpha$ into the span of $((p_a)^n : a \in \mathbb{F}^d)$. One such functional is $f \mapsto \Delta^\alpha f(0)$. If limits make sense in $\mathbb{F}$, then $f \mapsto D^\alpha f(0)$ is another. □