The standard inner product

The following bilinear form

$$\langle f, g\rangle := f(D)\overline{g}(0) = \sum_\alpha D^\alpha f(0)\overline{D^\alpha g(0)}/\alpha! \tag{1}$$

has become the standard inner product on $\Pi = \Pi(\mathbb{F}^d)$. When restricted to homogeneous polynomials (of the same degree), it is known as the **Bombieri inner product**. In other circles, it is known as the **Fisher inner product**. (It was supposedly already used by Calderon, in a discussion of harmonic polynomials.) It is evidently an inner product, given that the map

$$f \mapsto (D^\alpha f(0) : \alpha \in \mathbb{Z}_+^d)$$

is 1-1 and linear, and into finitely supported sequences, hence the infinite sum in (1) is well-defined.

This inner product is characterized by the fact that

$$\langle fg, h\rangle = \langle g, f(D)h\rangle, \quad f, g, h \in \Pi. \tag{2}$$

Indeed, the identity implies that

$$\langle ()^\alpha, ()^\beta\rangle = \beta(j)\langle ()^{\alpha-\mathbf{i}_j}, ()^{\beta-\mathbf{i}_j}\rangle,$$

hence, by induction, $\langle ()^\alpha, ()^\beta\rangle = 0$ whenever $\alpha(j) > \beta(j)$ for some j. By symmetry and induction, this implies that

$$\langle ()^\alpha, ()^\beta\rangle = \alpha!\langle ()^0, ()^0\rangle\delta_{\alpha,\beta},$$

hence, up to the positive scalar factor $\langle ()^0, ()^0\rangle$, the only inner product on Π satisfying (2) is the standard one. To be sure, the standard one evidently satisfies (2) since, after all,

$$(fg)(D)h = g(D)(f(D)h).$$

As Shayne Waldron points out, the standard inner product provides a quick proof of the fact that the space of homogeneous polynomials of degree n is spanned by ridge functions, i.e., functions of the form

$$(p_a)^n : x \mapsto (a^t x)^n, \quad a \in \mathbb{F}^d.$$

For, $p_a(D) = D_a$, hence any homogeneous polynomial q of degree n orthogonal to the span of ridge functions of degree n must satisfy

$$0 = \langle (p_a)^n, q\rangle = \langle 1, D_a^n q\rangle,$$

and, as $D_a^n q$ is a constant polynomial, it must be 0. Since $a \in \mathbb{F}^d$ is arbitrary, it follows that $q = 0$.

More generally, and then usually without the conjugation of the second argument, there is the standard pairing

$$\langle f, g\rangle := f(D)g(0) = \sum_\alpha D^\alpha f(0) D^\alpha g(0)/\alpha! = \sum_\alpha \hat{f}(\alpha)\alpha!\hat{g}(\alpha) \tag{3}$$

between $f =: \sum_\alpha \hat{f}(\alpha)()^\alpha$ in the space A_0 of formal power series and $g =: \sum_\alpha \hat{g}(\alpha)()^\alpha \in \Pi$. The resulting map $A_0 \to \Pi' : f \mapsto \langle f, \cdot\rangle$ is an invertible linear map, hence provides a representation of the dual Π' of Π.

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