Newton form

For any \( p \in \Pi_n \) and any scalar sequence \( c_1, \ldots, c_n \), there exists exactly one choice of coefficients \( a_0, \ldots, a_n \) so that
\[
p = \sum_{k=0}^{n} a_k \prod_{j=1}^{k} (-c_j).
\]
This is the Newton form with centers \( c_1, \ldots, c_n \) for \( p \), and its existence and uniqueness readily follows from the fact that the sequence
\[
(w_k := \prod_{j=1}^{k} (-c_j) : k = 0, \ldots, n)
\]
is linearly independent and in the \( n + 1 \)-dimensional linear space \( \Pi_n \), hence a basis for it.

The efficient evaluation of the Newton form is by nested multiplication aka Horner’s method. In this method, one takes advantage of the fact that \( w_k = (\cdot - c_k)w_{k-1} \), all \( k \), to write \( p \) in nested form,
\[
p = a_0 + (\cdot - c_1) (a_1 + (\cdot - c_2) (\cdots + (\cdot - c_{n-1}) (a_{n-1} + (\cdot - c_n)a_n) \cdots)),
\]
and then, for any particular scalar \( z \), obtains \( p(z) \) by evaluating this nested expression from the inside out. This gives the following algorithm.

\[
b_n := a_n; \text{ for } k = n-1:-1:0, \quad b_k := a_k + (z - c_{k+1})b_{k+1}; \text{ endfor}
\]
with \( b_0 \) equal to \( p(z) \).

More than that, since, in this way, \( a_k = b_k + (c_{k+1} - z)b_{k+1} \) for \( k < n \) and \( a_n = b_n \), it follows that
\[
p = \sum_{k=0}^{n} a_k w_k = b_0 + (\cdot - z) \sum_{j=1}^{n} b_j w_{j-1}.
\]
In other words, the sequence \( b_0, \ldots, b_n \) generated in Horner’s algorithm provides the coefficients in the Newton form for \( p \) with centers \( z, c_1, \ldots, c_{n-1} \).

This also makes clear why Horner’s method can be (and has been) thought of as an algorithm for dividing \( p \) by the linear polynomial \((\cdot - z)\).