Polynomial interpolation: existence, uniqueness

Let \( t_0, \ldots, t_n \) be an arbitrary scalar sequence and extend it, in any manner whatsoever, to an infinite scalar sequence. Then, by pagep111.pdf: “Newton form”, every \( f \in \Pi \) can be written in exactly one way in the form

\[
f = \sum_{k=0}^{\infty} a_k(f, t) \prod_{j<k}(\cdot - t_j)
\]

with the sum actually finite since \( a_k(f, t) = 0 \) for all \( k > \deg p \).

It follows that, with

\[
w_k := \prod_{j=0}^{k-1}(\cdot - t_j),
\]

the map

\[
P_n : f \mapsto \sum_{k=0}^{n} a_k(f, t)w_k
\]

is well-defined on \( \Pi \), linear, and maps \( \Pi \) into \( \Pi_n \). Further,

\[
f - P_nf = w_{n+1}q
\]

for some polynomial \( q \). If also \( p - g = w_{n+1}r \) for some \( g \in \Pi_n \) and some \( r \in \Pi \), then \( P_nf - g \) is a polynomial of degree \( \leq n \) and divisible by \( w_{n+1} \), hence must be zero.

It follows that \( P_n \) is a linear projector with range \( \Pi_n \), and \( P_nf \) is the unique polynomial of degree \( \leq n \) for which \( f - P_nf \) is divisible by \( w_{n+1} = (\cdot - t_0) \cdots (\cdot - t_n) \). But such divisibility is equivalent to the requirement that

\[
D^r(f - P_nf)(z) = 0, \quad 0 \leq r < \#\{0 \leq i \leq n : t_i = z\},
\]

i.e., to interpolation, in particular to repeated, or osculatory, or Hermite, interpolation in case of coincidences among the \( t_i \).

This suggests the standard extension of \( P_n \) to all sufficiently smooth \( f \), namely defining \( P_nf \) to be the unique polynomial of degree \( \leq n \) that satisfies the interpolation conditions (1).