The Kowalewski error formula

Apply the linear projector \( P_n : f \mapsto \sum_{j=0}^{n} \ell_j f(t_j) \), of polynomial interpolation at the distinct sites \( t_0, \ldots, t_n \), to the Taylor identity

\[
f(y) = \sum_{j \leq k} (y - x)^j D^j f(x)/j! + \int_x^y (y - t)^k D^{k+1} f(t) \, dt/k!
\]
as a function of \( y \), getting

\[
P_n f = \sum_{j \leq k} P_n (\cdot - x)^j D^j f(x)/j! + P_n F_k(x, \cdot, D^{k+1} f),
\]
with

\[
F_k(x, y, g) := \int_x^y (y - t)^k g(t) \, dt/k!.
\]

If now \( k \leq n \), then \( P_n (\cdot - x)^j = (\cdot - x)^j \) for all \( j \leq k \), hence then

\[
(1) \quad P_n f(x) = f(x) + P_n F_k(x, \cdot, D^{k+1} f)(x).
\]

To be sure, Kowalewski\(^{32a} \) (see pp21–24) only considers the case \( k = n \), and then gets

\[
f(x) = P_n f(x) + \sum_{j=0}^{n} \ell_j(x) \int_{t_j}^{x} (t_j - t)^n D^{n+1} f(t) \, dt/n!.
\]

But if \( k < n \), then, for any polynomial \( p \) of degree \( < n - k \), we can find a polynomial \( f \) of degree \( \leq n \) for which \( D^{k+1} f = p \), and for such \( f \), \( P_n f(x) = f(x) \), hence, by (1), \( P_n F_k(x, \cdot, p)(x) = 0 \). Thus,

\[
f(x) = P_n f(x) + \sum_{j=0}^{n} \ell_j(x) \int_{t_j}^{x} (t_j - t)^k (D^{k+1} f - p)(t) \, dt/k!, \quad \forall p \in \Pi_{<n-k}.
\]

With \( w_n := (\cdot - t_0) \cdots (\cdot - t_n) \), this error term can also be written

\[
w_n(x) \sum_{j=0}^{n} \int_{t_j}^{x} \frac{(t_j - t)^k}{Dw_n(t_j)(x - t_j)} (D^{k+1} f - p)(t) \, dt/k!, \quad \forall p \in \Pi_{<n-k}.
\]

Since

\[
DF_k(x, \cdot, g) = F_{k-1}(x, \cdot, g), \quad k \geq 0,
\]
it seems consistent to define
\[ F_{-1}(x, y, g) := g(y). \]
This also makes it easy to treat in the same way the linear map
\[ P_n^{(r)} : f \mapsto D^r P_n D^{-r} f \]
which reproduces \( \Pi_{n-r} \). For it,
\[
f(x) = P_n^{(r)} f(x) + \sum_{j=0}^{n} D^r \ell_j(x) \int_{t_j}^{x} (t_j - t)^n D^{n+1-r} f(t) \, dt / n!.
\]

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