Other B-spline recurrences

In Chakalov38a, Chakalov derives the following recurrence relation:

$$D \frac{M_{0,n}}{(\tau_n - \cdot)^{n-1}} = n \frac{M_{0,n-1}}{(\tau_n - \cdot)^n},$$

with

$$M_{i,j}(x) := (j - i) \Delta(\tau_i, \ldots, \tau_j)(\cdot - x)_{+}^{j-i-1}.$$

See also (3.4.6) in BojanovHakopianSahakian93.

As is pointed out in BoorPinkus03, differentiation of Chakalov’s recurrence gives the pretty formula

$$(n - 1)M_{0,n} + (\tau_n - \cdot)DM_{0,n} = nM_{0,n-1}$$

which occurs already in Morsche78 and is used heavily in MeinardusMorscheWalz95a and is, as shown in BoorPinkus03, equivalent to the standard B-spline recurrence relation. Note that the pretty formula is equivalent to Chakalov’s recurrence, and is an immediate consequence of the following special case

$$\Delta(t, s)(\cdot - x)f = (t - x)\Delta(t, s)f + \Delta(s)f$$

of the Leibniz formula, valid for any scalars $t$ and $x$, any sequence $s$, and any smooth enough $f$, by choosing $f = (\cdot - x)_+^{n-2} = -D_x(\cdot - x)_+^{n-1}/(n - 1)$, $s = (\tau_0, \ldots, \tau_{n-1})$, and $t = \tau_n$. Since the divided difference is a symmetric function of its sites, the sequence $\tau$ here is entirely arbitrary. Hence, the pretty formula holds for an arbitrary $\tau_n$, but with the warning that, if $\tau$ is not nondecreasing, then

$$\frac{\tau_n - \tau_0}{\max \tau - \min \tau}M_{0,n}$$

is the B-spline with knots $\tau_0, \ldots, \tau_n$ that integrates to 1.