Other B-spline recurrences

In Chakalov38a, Chakalov derives the following recurrence relation:

$$D\frac{M_{0,n}}{(\tau_n - \cdot)^{n-1}} = n\frac{M_{0,n-1}}{(\tau_n - \cdot)^n},$$

with

$$M_{i,j}(x) := (j-i)\,\Delta(\tau_i, \ldots, \tau_j)(\cdot - x)_+^{j-i-1}.$$

See also (3.4.6) in BojanovHakopianSahakian93.

As is pointed out in BoorPinkus03, differentiation of Chakalov's recurrence gives the pretty formula

$$(n-1)M_{0,n} + (\tau_n - \cdot)DM_{0,n} = nM_{0,n-1}$$

which occurs already in Morsche78 and is used heavily in MeinardusMorscheWalz95a and is, as shown in BoorPinkus03, equivalent to the standard B-spline recurrence relation. Note that the pretty formula is equivalent to Chakalov's recurrence, and is an immediate consequence of the following special case

$$\Delta(t, s)(\cdot - x)f = (t - x)\Delta(t, s)f + \Delta(s)f$$

of the Leibniz formula, valid for any scalars t and x, any sequence s, and any smooth enough f, by choosing $f = (\cdot - x)_+^{n-2} = -D_x(\cdot - x)_+^{n-1}/(n-1)$, $s = (\tau_0, \ldots, \tau_{n-1})$, and $t = \tau_n$. Since the divided difference is a symmetric function of its sites, the sequence τ here is entirely arbitrary. Hence, the pretty formula holds for an arbitrary τ_n, but with the warning that, if τ is not nondecreassing, then

$$\frac{\tau_n - \tau_0}{\max\tau - \min\tau}M_{0,n}$$

is the B-spline with knots $\tau_0, \ldots, \tau_n$ that integrates to 1.