The B-spline Gramian

Since

$$\Delta(t_0,\ldots,t_k)f = \int M(\cdot|t_0,\ldots,t_k)D^k f/k!,$$

with M the B-spline normalized to have integral 1, some integrals of the form

$$\int M(\cdot|t_0,\ldots,t_k)g$$

can (and certainly have been) computed from knowledge of divided differences, as long as it is easy to obtain $D^{-k}g$ and compute its divided difference. See pages004 B-spline¡/A¿ for moments or the Fourier transform of a B-spline.

In particular, with

$$N(y|t_j,\ldots,t_{j+h}) = (t_{j+h}-t_j)\,\Delta(t_j,\ldots,t_{j+h})(\cdot - y)^{h-1}$$

the B-splines (of order h) normalized to sum to 1, we compute

$$\begin{aligned}\int M(\cdot|t_i,\ldots,t_{i+k})&N(\cdot|t_j,\ldots,t_{j+h})\\ &= (-1)^k\binom{k+h-1}{k}(t_{j+h}-t_j)\,\Delta(t_i,\ldots,t_{i+k})_x\,\Delta(t_j,\ldots,t_{j+h})_y(x-y)_+^{k+h-1}.\end{aligned}$$

The special case $h=k$ appears already, in slightly different normalization, on p. 662 in

Jerome, J., Schumaker, L. L.; A note on obtaining natural spline functions by the abstract approach of Atteia and Laurent; SIAM J. Numer. Anal.; 5; 1968; 657–663;

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