The B-spline as a function of its knots

Since the normalized B-spline is given by

$$N_{0,k}(x) = N(x|t_0,\ldots,t_k) = (t_k - t_0)\Delta(t_0,\ldots,t_k)(\cdot - x)_+^{k-1},$$

it is easy to describe how it depends on its knots.

For example, it is well known that the derivative of a divided difference with respect to one of the points involved is simply the next-higher divided difference in which that point is repeated:

$$(d/ds)\Delta(s,t_1,\ldots,t_k) = \Delta(s,s,t_1,\ldots,t_k).$$

Consequently, (as was observed and exploited in Boor71), the derivative of N_{0k} with respect to its knot t_j is

$$(d/dt_j)N_{0k}(x) = (t_k - t_0)\Delta(t_j,t_0,\ldots,t_k)(\cdot - x)_+^{k-1},$$

if $0 < j < k$, while, e.g.,

$$(d/dt_0)N_{0k}(x) = -\Delta(t_0,t_0,t_1,\ldots,t_k)(\cdot - x)_+^{k-1}.$$

More generally, applying Hopf's identity (see Hopf26)

$$\Delta(t_0,\ldots,t_k) - \Delta(s_0,\ldots,s_k) = \sum_{j=0}^{k}(t_j - s_j)\Delta(t_0,\ldots,t_j,s_j,\ldots,s_k)$$

to the function $(\cdot - x)_+^{k-1} = -D_x(\cdot - x)_+^k/k$ under the assumption that $s_0 = t_0$ and $s_k = t_k$,

$$N(x|t) - N(x|s) = \sum_{j=1}^{k-1}\frac{s_j - t_j}{k}DN(x|t_0,\ldots,t_j,s_j,\ldots,s_k).$$

Already ter Morsche, in Morsche78, brings two proofs of the following B-spline identity

$$N(x|t) = \frac{x - t_i}{k-1}DN(x|t) + N(x|t(\backslash i)),$$

valid for any knot sequence $t = (t_0,\ldots,t_k)$ and $0 < i < k$ (and with $t(\backslash i)$ the sequence obtained from t by omitting its ith entry). Perhaps the quickest proof is via the special case

$$\Delta(t)(\cdot - x)f = (t_i - x)\Delta(t)f + \Delta(t(\backslash i))f$$

of Leibniz' formula, with the choice $f = (\cdot - x)_+^{k-2} = -D_x(\cdot - x)_+^k/(k-1)$. This also shows that, e.g.,

$$N(x|t) = \frac{x - t_k}{k-1}DN(x|t) + \frac{t_k - t_0}{t_{k-1} - t_0}N(x|t(\backslash k)).$$

Also, it is pointed out in BoorPink04 that the above identity can be obtained by differentiating Chakalov's recurrence relation (see, e.g., (3.4.6) in BHS93, or (11) in BoorPink04)

$$D\frac{M(\cdot|t)}{(t_k - \cdot)^{k-1}} = k\frac{M(\cdot|t(\backslash k))}{(t_k - \cdot)^k}.$$