The perfect B-spline

The perfect B-splines, i.e., B-splines whose piecewise constant derivative is absolutely constant, make their first appearance in Schoenberg71d, as the solution to an optimal control problem.

These are, up to a scalar multiple, the B-splines

$$M(x|t_0, \ldots, t_k) : x \mapsto k \Delta(t_0, \ldots, t_k)(x - x)^{k-1}$$

whose knots $t_j := \cos((k-j)\pi/k)$, $j = 0:k$, are the extrema of the Chebyshev polynomial of degree $k$, hence (e.g., Rivlin74b or [http://mathworld.wolfram.com/ChebyshevPolynomialoftheSecondKind/]

$$\omega(x) := \prod_j (x - t_j) = (x^2 - 1)U_{k-1}(x)/2^{k-1},$$

with $U_{k-1}(\cos \theta) = \sin(k\theta)/\sin(\theta)$, i.e., $U_n$ is a Chebyshev polynomial of the second kind, while, by [divided difference: basic formula],

$$\Delta(t_0, \ldots, t_k) = \sum_j \Delta(t_j)/(D\omega)(t_j).$$

Now observe that $D\omega(x) = 2xU_{k-1}(x) + (x^2 - 1)DU_{k-1}(x)$. So, for $0 < j < k$, $2^{k-1}D\omega(t_j) = (t_j^2 - 1)DU_{k-1}(t_j)$, hence, with $x := t_j =: \cos \theta$ and $y := \sin \theta$,

$$2^{k-1}D\omega(t_j) = -y^2(k \cos(k\theta)/y) \star 1/(-y) = (-)^j,$$

while, for $j = 0$ or $k$, $2^{k-1}D\omega(t_j) = 2t_jU_{k-1}(t_j) = 2(-)^j$.

Hence,

$$2^{k-1}M(x|t_0, \ldots, t_k)/k = (-1 - x)^{k-1}/2 + \sum_{j=1}^{k-1} (-)^j(t_j - x)^{k-1} + (-1)^k(1 - x)^{k-1}/2,$$

showing $|D^{k-1}M| = k/2^k$ on its support.