local interpolation

In BirkhoffBoor64, there is an unsupported (and actually misprinted) claim about the minimum polynomial degree achievable by a local interpolation scheme to data values (y_j) at a corresponding strictly increasing sequence (x_j) of interpolation sites by $C^{(m)}$ pp functions with breaks only at the x_j.

Write such an interpolant as a linear combination of B-splines of order k. Since it is to be in $C^{(m)}$, each x_j may appear $k-(m+1)$ times in the corresponding knot sequence t. We are now looking for the smallest M for which the linear system

$$(1) \qquad \sum_{x_{r-M}\le t_j<t_{j+k}\le x_{r+M}} B(x_i|t_j,\ldots,t_{j+k})a(j) = \delta_{ir}, \quad i=r-M+1,\ldots,i+M-1,$$

has a solution, and cannot hope for one unless $k>m+1$. This is a linear system of $2M-1$ equations in $(2M+1)(k-(m+1))-k = 2Mk-(2M+1)(m+1)$ unknowns, hence having nontrivial solutions requires $2M-1\le 2Mk-(2M+1)(m+1)$ or $m\le 2M(k-m-2)$), hence, for $m>0$, we need

$$k>m+2.$$

On the other hand, if $k>m+2$, then we can choose

$$M := \lceil m/(2(k-m-2)))\rceil,$$

and, with such a choice, know that, after setting in (1) some unknowns to zero, we obtain a square linear system that satisfies the Schoenberg-Whitney conditions (i.e., has nonzero diagonal entries), hence is invertible.

Simplest examples are:

$m=0$ $L_r = B(\cdot|x_{r-1:r+1})$ i.e., broken line interpolation ($k=2$)

$m=1$ $L_r = B(\cdot|x_{r-1},x_{r-1},x_r,x_r,x_{r+1}) + B(\cdot|x_{r-1},x_r,x_r,x_{r+1},x_{r+1})$, ($k=4$, $M=1$)

$m=2$ $L_r = B(\cdot|x_{r-1},x_{r-1},x_r,x_r,x_{r+1},x_{r+1})$, ($k=5$, $M=1$)

But none of these reproduces more than just Π_1. If we want to reproduce Π_s for $s>1$, we have to work harder, and M has to increase.

Have a look at Rutishauser's Glatte Interpolation which achieves a local interpolant in $C^{(r)}$ by piecewise polynomials of degree $2r+1$ with breaks only at the data sites, and the polynomial piece on $[x_i \,..\, x_{i+1}]$ only depending on the $2r+2$ data values $y_{i-r},\ldots,y_{i+1+r}$, and reproducing all polynomials of degree $\le 2r$.

last updated 12may09