Rutishauser's "Glatte Interpolation"

Rutishauser's paper [R] (some have claimed that its results can already be found in [J]) contains the following construction, worked out explicitly there for data $P_i := (x_i, y_i)$, with $x_i = x_0 + ih$, all i. Rutishauser obtains an interpolant q to such data that is piecewise polynomial of degree $\le 2r+1$ with breaks at the x_i and has s continuous derivatives and, on the interval $(x_i \,..\, x_{i+1})$, depends only on the $2r+2$ data points P_j, $j \in i + R$ with $R := (-r){:}(r+1)$.

Explicitly, on the interval $(x_i \,..\, x_{i+1})$, the interpolant is of the form

$$q_{i+1/2}(x) := p_{i+1/2}(x) + \psi(t(x))\binom{t(x)+r-s}{2r-2s+2}\delta^{2r+1}_{i+1/2}y, \tag{1}$$

with $p_{i+1/2}$ the unique polynomial of degree $\le 2r+1$ that matches the $2r+2$ data points P_j, $j \in i+R$, with $t(x) := (x - x_i)/h$, with ψ a certain polynomial of degree $2s-1$ and odd around $1/2$, and with $\delta^{2r+1}_{i+1/2}y$ the central difference of the y_j, $j \in i+R$. Moreover, Rutishauser proves that, for $s = r$, his construction provides the unique solution of this form, i.e., as the standard central local polynomial interpolant modified locally in such a way that the resulting pp function is in $C^{(r)}$.

Here is a quick argument for Rutishauser's claim: Directly from (1), we conclude that the construction reproduces any polynomial of degree $\le 2r$ (as the $2r+1$st difference for such data is zero). Rutishauser quite explicitly makes this point, namely that he achieves the smoothness at the loss of only one order of approximation. Further, the ith piece, $q_{i+1/2}$, depends (linearly) only on the data P_j, $j \in i+R$. This means that the matching information, i.e., the numbers $q^{(j)}_{i-1/2}(x_i) = q^{(j)}_{i+1/2}(x_i)$, $j = 0{:}s$, can only depend on the information common to both $q_{i-1/2}$ and $q_{i+1/2}$, i.e., on P_j, $j \in i + (-r){:}r$. Hence, if we also demand that $s = r$, then, necessarily,

$$q^{(j)}(x_i) = p_i^{(j)}(x_i), \quad j = 0{:}r,$$

with p_i the unique polynomial of degree $\le 2r$ matching the P_j, $j \in i + (-r){:}r$. But once we know q in this way $r+1$-fold at x_i and x_{i+1}, we know it on $(x_i \,..\, x_{i+1})$ since, there, it is just a polynomial of degree $\le 2r+1$.

Note that this argument actually works for arbitrary spacing of the x_i, though one may have to work harder to get Rutishauser's explicit formula for this general case.

Smooth local piecewise polynomial interpolation, particularly for uniformly spaced data sites, was an interesting topic in the middle of the last century; see, e.g., Schoenberg's fundamental 1946 paper [S] which also summarizes the earlier work of the actuarians such as [J].

[J] W. A. Jenkins (1926), "Osculatory interpolation: new derivation and formulas", *Record Amer. Inst. Actuar.* **15**, 87–98.

[R] Heinz Rutishauser (1960), "Bemerkungen zur glatten Interpolation", *Z. Angew. Math. Phys.* **11**, 508–513.

[S] I. J. Schoenberg (1946), "Contributions to the problem of approximation of equidistant data by analytic functions, Part A: On the problem of smoothing or graduation, a first class of analytic approximation formulas", *Quart. Appl. Math.* **4**, 45–99.