Lec 13: Symbolic ODE Solutions

- Exam 3 is Not Cumulative (Mods 10-15)
  - Day: Wednesday, Dec 21st
  - Time: 10:05am-12:05pm
  - Room: 19 Ingraham Hall

- Homeworks
  - H7 is due before 9am Thursday, 12/08/11
  - H8 is due before 4pm Thursday, 12/15/11

- Today: Symbolic Solutions to ODEs
  - Bring laptop with scratch.mit.edu installed to lab this week

- Next Week: Numeric Solutions to ODEs
Scratch.MIT.edu

- Scratch is a free development environment targeted at middle-school aged students.
- It allows users to graphically create animations, games, and applications.
- Interested Students:
  - Install Scratch on your personal laptop
  - Bring laptop to lab this week.
  - Create an account on scratch.mit.edu and you will be able to post your creations
What is the result of solving a set of three linear equations for three unknowns?

\[
sol := \text{solve}( \{eq1, eq2, eq3\}, \{x, y, z\} )
\]

- an expression
- an equation
- a set of expressions
- a set of equations
- a list of expressions
- a list of equations
- a sequence of expressions
- a sequence of equations
What is an ODE?

- An equation that describes the relationship between a function and its derivative.
- A common problem in engineering. Often, we know the rate of change without knowing the function that produces that rate of change.
- Dynamic System Examples: (change wrt time)
  - Radioactive Decay
  - Diffusion and Heat Transfer
  - System Controls
  - Population Dynamics
Linear First Order ODEs

A linear first order ODE contains only the first derivative and linear function terms (no squared function terms).

\[
\text{Linear First Order ODE} \quad \Rightarrow \quad \text{linearode} := \text{diff}(y(t), t) = \alpha y(t) + \beta
\]
Which of these are Linear First Order ODEs?

\[
\frac{dy(t)}{dt} = 3y(t)
\]

\[
\frac{dp(t)}{dt} = \alpha p(t)^{\frac{1}{2}}
\]

\[
\frac{dq(t)}{dt} = \frac{1}{q(t)}
\]

\[
\frac{dr(t)}{dt} = 3\sin(t)^2 r(t)
\]

\[
\frac{ds(\theta)}{d\theta} = \sin(s(\theta))
\]
Solving Linear First Order ODEs

By Hand

0. To solve this ODE by hand.

\[ \frac{d}{dt} f(t) = \alpha f(t) \]

1. Rewrite ODE in this Form: \( f'(t) + p(t)f(t) = g(t) \)

\[ \frac{d}{dt} f(t) - \alpha f(t) = 0 \]

Note: In the above example, \( p(t) = -\alpha \) and \( g(t) = 0 \)

2. Compute \( u(t) \) as \( \exp(\int p(t) \, dt) \)

\[ u(t) = e^{-\alpha t} \]

3. Compute \( f(t) \) as \( 1/u(t) \cdot (\int u(t) g(t) \, dt) + C \)

\[ > f\text{sol} := \frac{1}{u(t)} \cdot (\int u(t) g(t) \, dt) + C \]

\[ f\text{sol} := \frac{C}{e^{-\alpha t}} \]

\[ > f\text{sol} := \text{simplify}(f\text{sol}) \]

\[ f\text{sol} := C e^{\alpha t} \]
Check Solution

- Take derivative of the solution function and plug that result and the function into the ODE.
- Should produce the equivalent ODE.

\[
\frac{d}{dt} f(t) - \alpha f(t) = 0
\]

4. Check the solution by plugging terms into ODE.
\[
> \text{checkDiff} := \text{diff}(fissol, t) - \text{alpha} \cdot fissol = 0
\]
\[
\text{checkDiff} := 0 = 0
\]
\[
> \text{check2} := \text{subs}(\text{diff}(f(t), t) = \text{diff}(fissol, t), f(t) = fissol, \text{gen})
\]
\[
\text{check2} := 0 = 0
\]
or Solve Symbolically in Maple?

1) Construct the ODE
2) Enter the ODE symbolically
3) Use `dsolve` to solve the ODE symbolically
4) Substitute values for symbolic constants
5) Explicitly plot the solution function
Solve Symbolically without and with an Initial Condition value

Symbolically Solve an ODE

\[ \text{ode} := \text{diff}(f(t), t) = \alpha f(t) + \beta \]

\[ \text{ode} := \frac{d}{dt} f(t) = \alpha f(t) + \beta \]

Without an Initial Condition

\[ \text{sol} := \text{dsolve}(\text{ode}, f(t)) \]

\[ \text{sol} := f(t) = -\frac{\beta}{\alpha} + _C1 \ e^{\alpha t} \]

With and Initial Condition

\[ \text{solInitCond} := \text{dsolve} \left( \{ \text{ode}, f(0) = 3 \}, f(t) \right) \]

\[ \text{solInitCond} := f(t) = -\frac{\beta}{\alpha} + \left( 3 + \frac{\beta}{\alpha} \right) e^{\alpha t} \]

set including ode and initial condition equations
Find Steady-State of an ODE System

- Set the derivative to zero
- Symbolically (algebraically) solve for the function term.

```
Find the Steady-State
> ssEqn := subs(diff(f(t), t) = 0, ode)
    ssEqn := 0 = \alpha f(t) + \beta
> ssSol := solve(ssEqn, f(t))
    ssSol := \frac{-\beta}{\alpha}
```
Plot a Symbolic Solution to an ODE

- Solve the ODE with initial conditions, substitute for any unknowns, and plot the \( \text{rhs()} \) of the solution equation.

**Plot the Solution**

\[
\begin{align*}
\text{solsub} & := \text{subs}\left(\alpha = \frac{2}{5}, \beta = \frac{1}{5}, f_0 = 3, \text{solInitCond}\right) \\
\text{solsub} & := f(t) = -\frac{1}{2} + \frac{7}{2} e^{\frac{2}{5}t} \\
\text{plot}(\text{rhs}(\text{solsub}), t = 0..15)
\end{align*}
\]
Next Week

- Using Maple to find Numeric Solutions to Ordinary Differential Equations
**intersectplot( )**

- Plots the intersection of two 3D surfaces.
- $e2 := 2x^2 + (y-3)^2 + z^2 = 3$
- $e3 := 2x + 2y + z = 4$

```
To show the intersection of these two surfaces.
> intersectplot(eq2, eq3, x=-2..2, y=1..6, z=-3..3, axes=box,
    scaling=constrained)
```

![Graph showing the intersection of two 3D surfaces](image)