CS 880: Advanced Complexity Theory	1/25/06
Lecture 2: Existence of Expanders	
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Last time we saw the original definition of an expander: a graph $G = (V, E)$ is (K, c)-expanding if for all subsets $A \subseteq V$ with $|A| \leq K$ we have $|\Gamma(A)| \geq c \cdot |A|$. In today's lecture we use the probabilistic method to prove that such graphs exist (though the proof we give is not constructive, later in the course we will give some very nice, efficient methods to explicitly construct expander graphs). We will also consider a spectral characterization of expander graphs that is equivalent to the original definition. This alternate characterization is very useful for many applications because it immediately leads to the *expander mixing lemma*, as we shall see in later lectures.

1 Proof of existence

We would like to show that most d-regular graphs are highly expanding. How large does d have to be for this to make sense? We will discuss some cases assuming there are no self loops; this assumption can easily be generalized. If we consider $d = 1$, then the graph G must be a matching (pairs of vertices joined by an edge). We would like the parameter K to be some constant fraction of N, the number of vertices, and a little thought shows that for such K there cannot be a constant $c > 1$ such that G is (K, c)-expanding: for any set $A \subseteq V$ the neighbor set $\Gamma(A)$ has exactly the same size as A ($|\Gamma(A)| = |A|$), and therefore A doesn't expand at all. So $d = 1$ is no good.

What about $d = 2$? Then G is the union of disjoint cycles. But once again, for K that is a constant fraction of N, we can choose a bad A: let A be a path on one of the cycles, so that $|\Gamma(A)| \leq |A| + 2$. For $|A|$ of more than constant size, this will not be enough increase to satisfy the expander property. So $d = 2$ is also not good.

It turns out that $d \geq 3$ is where the graph G can have enough complexity so that things get interesting. We will prove this in a moment, but first let us consider appropriate bounds for c. We want $c > 1$, but we must have $c \leq d$, since each vertex can contribute at most d neighbors to the neighbor set $\Gamma(A)$. Furthermore, we can always set A to be a non-self-intersecting path in G, and then a simple calculation tells us that $\Gamma(A) \leq |A| + (d-2)|A| + 2$. Asymptotically, this gives us $c \leq d - 1$. We now present the main theorem of this section.

Theorem 1. *For any degree d and real number $b > 1$, there exists $\alpha(b, d) > 0$ such that*

$$\Pr[d\text{-regular graph on } N \text{ vertices is } (\alpha \cdot N, d - b) \text{ expanding}] \geq .99$$

(Notice that in the above theorem the restriction $d \geq 3$ is implied, since the statement is only interesting when $c = d - b > 1$, and we have $b > 1$.) We will actually prove the result for a slightly different case, where instead of considering a randomly chosen d-regular graph, we consider a bipartite graph with vertex sets L and R, $|L| = |R| = N$, and where every vertex in L has exactly d neighbors which are chosen uniformly at random from R.

Proof (for bipartite case): We use the probabilistic method to upper bound the probability that a random bipartite graph is not sufficiently expanding: we will calculate the probability of an elementary bad event occurring, and then use a union bound to upper bound the probability that any of these bad events occur. The elementary bad event we consider is the following (for $K = \alpha \cdot N$ to be determined later): For $A \subseteq L$, $|A| \leq K$, we have $|\Gamma(A)| < (d-b)|A|$; that is, the desired expander condition is violated.

Fix $k \leq K$ and consider $A \subseteq L$ with $|A| = k$. Then the probability that A violates the desired expander condition is the probability that $|\Gamma(A)| < (d-b)k$. If we think of ordering the dk neighbor choices we make for A (d for each of the k vertices in A), then we see that the probability of this bad event is the same as the probability that more than bk of the dk random neighbor choices coincide with previous choices, giving

$$|\Gamma(A)| < dk - bk = (d-b)k.$$

We can therefore write

$$\begin{aligned}
\Pr[\ |\Gamma(A)| < (d-b)k] &= \Pr[\text{more than } bk \text{ choices collide with earlier choices}] \\
&\leq \Pr[\text{at least } bk \text{ choices collide with earlier choices}] \\
&\leq \binom{dk}{bk}\left(\frac{dk}{N}\right)^{bk}
\end{aligned}$$

where the last line follows because there are $\binom{dk}{bk}$ ways to choose bk colliding choices from dk choices, and each such collision occurs with probability no more than dk/N for each of the bk choices. This is certainly an overestimate, but it is good enough for our purposes.

We now use a union bound, along with the approximation

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k,$$

to upper bound the probability that there exists such a bad A of size k.

$$\begin{aligned}
\Pr[\text{there exists a bad } A \text{ of size } k] &\leq \binom{N}{k} \cdot \binom{dk}{bk}\left(\frac{dk}{N}\right)^{bk} \\
&\leq \left(\frac{eN}{k}\right)^k \left(\frac{ed}{b}\right)^{bk} \left(\frac{dk}{N}\right)^{bk} \\
&= \left(e^{b+1}\left(\frac{d^2}{b}\right)^b \left(\frac{k}{N}\right)^{b-1}\right)^k \\
&\leq \left(\underbrace{e^{b+1}\left(\frac{d^2}{b}\right)^b \alpha^{b-1}}_{(*)}\right)^k
\end{aligned}$$

Note that we can make the $(*)$ term arbitrarily small - say, less than β - by making the constant α sufficiently small. This is because the restriction $b > 1$ means α will always have a strictly positive exponent in the term above. A final union bound over all sizes $k \leq K$ gives us the desired result:

$$\Pr[\text{there is some bad } A,\ |A| \leq K] \leq \sum_{k=1}^{\alpha \cdot N} \beta^k$$

This geometric sum will converge to some constant less than .01 for small enough β. □

Notice that our random construction doesn't give a d-regular graph, because the vertices in R may have degree other than d. This problem can be fixed by choosing d perfect matchings, but we will not go into detail here. We also note that it is not currently known how to derandomize this process in order to explicitly construct an expander; there are, however, other ways to accomplish an explicit construction, as we shall see in another lecture.

2 The spectrum of a d-regular graph

In graph theory it is often convenient to work with the adjecency matrix a graph. Expanders are no exception: there is a connection between the vertex expansion of a graph and the spectrum of it's adjacency matrix. We give a brief overview of some basic but important properties of the spectrum of a graph (when we write the spectrum of a graph, we implicitly mean the spectrum of its adjacency matrix). Let us fix A, the adjacency matrix of a d-regular graph $G = (V, E)$ with $|V| = N$; and let A be normalized (its rows sum to 1).

2.1 General properties of the spectrum of G

- The spectra of any graph is real. The reason is that since A is real and symmetric, it has only real eigenvalues.
- All eigenvalues have at most norm 1. In order to see this is true we will need a property true for all matrices over the complex numbers: all eigenvalues are within the union of certain discs in the complex plane, namely for every row i, there is a disc centered at A_{ii} with radius $\sum_{j \neq i} |A_{ij}|$. To see this is true consider, $Ax = \lambda x$ with λ any eigenvalue corresponding to

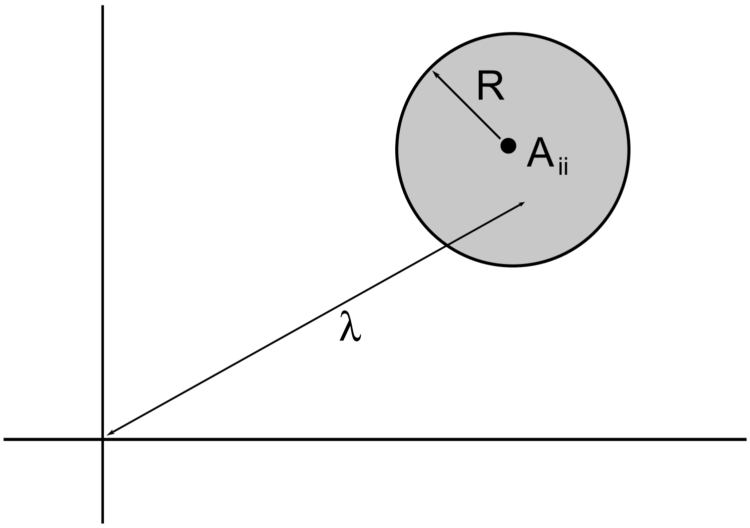

Figure 1: Every eigenvalue λ is withing a certain disc of radius $R = \sum_{j \neq i} |A_{ij}|$ centered at A_{ii}.

eigenvector $x = (x_1, \cdots, x_N)$. Suppose x_i is the largest component of x in absolute value.

Since $x \neq 0$, surely $x_i \neq 0$. Now some algebra shows,

$$\sum_{j=1}^{N} A_{ij} x_j = \lambda x_i,$$
$$\sum_{j=1}^{N} A_{ij} \frac{x_j}{x_i} = \lambda,$$
$$\sum_{j \neq i} |A_{ij}| \, |\frac{x_j}{x_i}| \geq |A_{ii} - \lambda|,$$
$$\sum_{j \neq i} |A_{ij}| \geq |A_{ii} - \lambda|,$$

or in other words, every λ is within a distance $\sum_{j \neq i} |A_{ij}|$ of some A_{ii} in the complex plane. Since in our case, the sum of all elements in a row is 1, it follows that all λ are less than 1.

- 1 is always an eigenvalue with corresponding eigenvector $u = \frac{1}{N}(1, 1, \cdots, 1)$. It is possible that the multiplicity of the 1 eigenvalue is greater then one. It is a good exercise to show that this happens if and only if the graph is disconnected.

- -1 can also be an eigenvalue in the spectrum. Again it is a good excersise to show that this happens if and only if the underlying graph is bipartite.

Definition 1. *We define λ_2 to be the largest eigenvalue in absolute value corresponding to an eigenvector which is not in **span**(u). This definition is equivalent to defining,*

$$\lambda_2 = \max_{x \perp u, ||x||_2 = 1} ||Ax|| = \max_{x \perp u, ||x||_2 = 1} x^\top Ax. \tag{1}$$

We already know that all eigenvalues have an absolute value at most 1 so this certainly holds for λ_2. Moreover, by the properties we stated above, we can state that $\lambda_2 < 1$ if and only if G is connected and not bipartite.

Definition 2. *A graph G has spectral gap $\geq 1 - \lambda$ iff $\lambda_2 \leq \lambda$.*

The spectral gap is key in the connection between the vertex expansion of a graph and it's spectrum: roughly we will prove that a family of graphs is (K, c)-expanding iff it has spectral gap $\geq 1 - \lambda$ for some $\lambda < 1$.