CS 880: Advanced Complexity Theory 2/6/06

Lecture 6: Zig-Zag Product

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In the previous lecture, we saw two different constructions for expander graphs. The Gabber-Galil construction has the benefit of being very explicit, while the LPS&M construction has a spectral gap that is asymptotically optimal. Both of the constructions are explicit in the sense that there is an algorithm that takes the parameters as input and efficiently (in one pass) generates the graph. However, there is not a simple intuition for either of the graph families as to why they have the expansion properties they do.

Today, we will introduce the zig-zag product. This is a key element in an expander graph construction that we will see in the next lecture. The spectral gap of this graph family is not as good as the spectral gap of the constructions we saw last lecture; and the construction is iterative, meaning we will be sacrificing some of the explicitness. However, there is a simple intuitive idea for why the graphs have a spectral gap. Applications of the zig-zag based expander include the simple proof of the PCP theorem and the log-space algorithm for undirected connectivity.

1 Replacement Product

We begin by introducing a product which is closely related to the zig-zag product, the replacement product. The replacement product is an operation that takes two graphs G, H as input and produces a new graph $G \textcircled{r} H$ as output. Intuitively, we will take each vertex v of G and replace it with a copy of H, which we call the "cloud" of v. We then connect clouds corresponding to u and v if $(u, v) \in E(G)$ by selecting unique vertices in these clouds to connect. To do this, we require that the number of vertices in H is equal to the degree of the vertices in G. More formally, the definition is as follows:

Definition 1 (Replacement Product)**.** *Let G and H be graphs such that $d(G) = |\mathcal{V}(H)|$. Then $G \textcircled{r} H$ is defined as follows:*

- *The graph $G \textcircled{r} H$ has $|\mathcal{V}(G)| \cdot |\mathcal{V}(H)|$ vertices. For each pair of vertices $v \in G, i \in H$, there is a vertex (v, i) in $G \textcircled{r} H$.*
- *For each v, there is an edge $((v, i), (v, j))$ if there is an edge $(i, j) \in H$.*
- *For each vertex in G, fix a numbering of its neighbors. For each edge $(v, w) \in G$, there is an edge $((v, i), (w, j))$ in $G \textcircled{r} H$ where w is the i^{th} neighbor of v, and v is the j^{th} neighbor of w.*

For each copy of H that is inserted into G, it is associated with a vertex v from G. We call this "copy" of H the cloud of v.

Figure 1 shows the replacement product graph for $K_5 \textcircled{r} C_4$.

Properties of the replacement product include:

- $|\mathcal{V}(G \textcircled{r} H)| = |\mathcal{V}(G)||\mathcal{V}(H)| = |\mathcal{V}(G)|d(G)$.

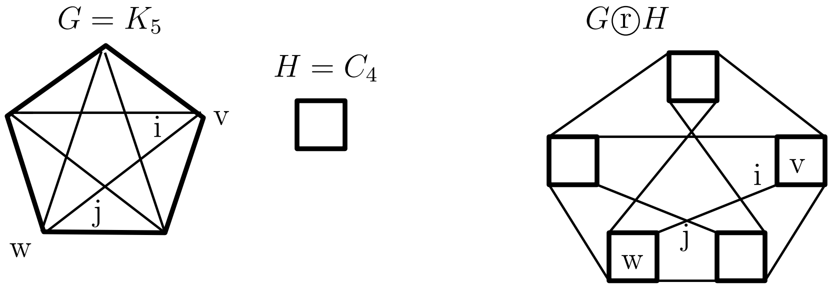

Figure 1: Replacement product of $G = K_5$ with $H = C_4$. The i^{th} neighbor of v and j^{th} neighbor of w are labeled in G, and the corresponding edge from the "cloud" of v to the "cloud" of w is labeled in $G \textcircled{r} H$.

- $d(G \textcircled{r} H) = d(H) + 1$.
- There is a function f such that $\lambda_2(G \textcircled{r} H) \leq f(\lambda_2(G), \lambda_2(H), d(H))$ where $f(\lambda_2(G), \lambda_2(H), d(H)) < 1$ when $\lambda_2(G) < 1$ and $\lambda_2(H) < 1$.

The important properties of the replacement product are that the number of vertices goes up, the the degree is low, and the spectral gap doesn't deteriorate too much. The intuition behind this last item is easier to explain in terms of the zig-zag product, which we introduce next.

2 Zig-Zag Product Definition

The zig-zag product is constructed with the same vertex set as the replacement product while its edge relation is defined such that one step of a random walk from vertex (v, i) is equivalent to the following three steps — (i) One step of a random walk on H starting from i in the cloud of v, (ii) Use the edge from $G \textcircled{r} H$ to switch to a different cloud, say w, and (iii) one step of a random walk on H in cloud w. More formally, we have:

Definition 2 (Zig-zag product)**.** *Given graphs G and H such that $d(G) = |\mathcal{V}(H)|$. The vertex set of $G \textcircled{z} H$ is the same as that for $\textcircled{r}$: there is a vertex (v, i) for each pair of vertices v from G and i from H.*

For vertices (v, i) and (w, j) there is an edge between them in $G \textcircled{z} H$ if there is a path in $G \textcircled{r} H$ such that there are vertices $i', j' \in V(H)$ such that:

- *$(i, i') \in E(H)$,*
- *$(j', j) \in E(H)$, and*
- *$((v, i'), (w, j'))$ is an edge in $G \textcircled{r} H$.*

Properties of the Zig-Zag product include:

- $\mathcal{V}(G \textcircled{z} H) = \mathcal{V}(G) \times \mathcal{V}(H)$.
- $d(G \textcircled{z} H) = (d(H))^2$. Note that this is worse than in the replacement product, but will still be constant when $d(H)$ is constant.

- There is a function g such that $\lambda_2(G \textcircled{z} H) \leq g(\lambda_2(G), \lambda_2(H))$ where $g(\lambda_2(G), \lambda_2(H)) < 1$ when $\lambda_2(G) < 1$ and $\lambda_2(H) < 1$.

Notice that the edge set of $G \textcircled{z} H$ is a subset of the edge set of $(G \textcircled{r} H)^3$. This means the expansion of $(G \textcircled{r} H)^3$ is at least as good as the expansion of $G \textcircled{z} H$, so that proving a spectral gap for the latter also shows a spectral gap for the former. Also, notice although we give the definition of the zig-zag product of graphs, it easily extends to multigraphs as well.

For the intuition of why this construction gives a spectral gap, we will need to define various notions of entropy.

3 Entropy

Given a distribution π, we can define the entropy of the distribution in various ways.

Definition 3 (standard entropy)**.** $H(\pi) = -\Sigma_x \pi(x) \log(\pi(x)) = E_x[-\log(\pi(x))]$.

Definition 4 (min-entropy)**.** $H_\infty(\pi) = -\max_x\{\log(\pi(x))\} = -\log(\|\pi\|_\infty)$.

Definition 5 (Renyi entropy)**.** $H_2(\pi) = -\log(\Sigma_x((\pi(x))^2)) = -\log(\|\pi\|_2^2)$. *Recall that* $\|\pi\|_2^2$ *is the collision probability of* π*.*

It is an exercise to show that these definitions of entropy relate in the following way for any distribution π:

$$H_\infty(\pi) \leq H_2(\pi) \leq H(\pi).$$

4 Zig-Zag Product Spectral Gap

We would like to show that there is a spectral gap on $G \textcircled{z} H$ when there is a gap for both G and H. Let A be the normalized adjacency matrix of $G \textcircled{z} H$. We use the following two properties where π is a distribution on the vertices of $G \textcircled{z} H$:

1. $H_2(A\pi) \geq H_2(\pi)$. This is a strict inequality when $\lambda_2(A) < 1$ and $\pi \neq u$. In other words, performing a step of a random walk increases the entropy of a distribution. We proved this earlier in terms of collision probability.

2. $H(\pi) = H(v) + E_v[H(i|v)]$. In this equation, $H(v)$ refers to the entropy of the marginal distribution of π on the vertices of G, and $H(i|v)$ is the conditional distribution of π within the cloud of v.

If we can show that the entropy for any distribution $\pi \neq u$ increases after a step in the zig-zag product graph, the first property tells us that $\lambda_2(G \textcircled{z} H) < 1$ as desired. However, we will only be able to argue this by switching back and forth between definitions of entropy. Nonetheless, this provides intuition why there should be a spectral gap.

Claim 1. *For any* $\pi \neq u$ *on* $\mathcal{V}(G) \times \mathcal{V}(H)$*,* $H(A\pi) > H(\pi)$*, where* A *is the normalized adjacency matrix of* $G \textcircled{z} H$*.*

Proof. An edge in the zig-zag product arises from three stages: one step random walk within a cloud, one step to another cloud, and then one step random walk within that cloud. Let us use equality (2) above to consider how the entropy of a distribution is affected by each step.

a) For the first step, we stay in the same cloud, so $H(v)$ is unaffected. However, item (1) tells us that $H(i|v)$ will go up when $i|v$ is not the uniform distribution, and will stay the same when $i|v$ is the uniform distribution.

b) For the second step, we are performing one deterministic move (there is only one possible edge to take out of a cloud from a given point in the cloud). This means the total entropy remains the same after this step. Another way to think of this is that the probability mass has been shifted around the graph, but the sum will remain the same.

c) The third step is similar to the first: $H(v)$ remains unchanged, while $H(i|v)$ goes up if $i|v$ at that point is not the uniform distribution.

Now let us consider two cases for how the entropy is affected by a step in the zig-zag product graph:

Case 1: For at least one v, the distribution of $i|v$ is not uniform. In this case, the first step from a point in the cloud of v will increase the entropy $H(i|v)$, while the second and third steps cannot decrease it. Therefore, $E_v[H(i|v)]$ increases and $H(v)$ doesn't decrease, so by item (2) the total entropy goes up.

Case 2: The distribution of $i|v$ is uniform for all v. In this case, neither the first or second steps affect total entropy. However, when $i|v$ is uniform, we can view these steps as implementing one step of a random walk on the graph G. By (1), this increases the marginal entropy $H(v)$. Because $H(v)$ goes up but the total entropy stays the same, by item (2) we must have that $E_v[H(i|v)]$ goes down in the second step. Therefore, there must be some v so that $i|v$ is no longer uniform, so that $E_v[H(i|v)]$ goes up in the third step.

In either case, the entropy goes up, so $H(A\pi) > H(\pi)$. $\square$

Combining the claim with the property mentioned about H_2 above, this leads us to believe that there is a spectral gap for $G \text{ⓩ} H$. Note the problem with this argument for expansion: we are using two different notions of entropy interchangeably.

Next time, we will give a formal proof that the zig-zag product maintains a spectral gap.

Theorem 1. *$\lambda_2(G \text{ⓩ} H) \leq \lambda_2(G) + \lambda_2(H) + (\lambda_2(H))^2$.*

It can also be shown that:

Theorem 2. *$\lambda_2(G \text{ⓩ} H) \leq \lambda_2(G) + \lambda_2(H)$.*

Notice that this is still not enough to show that $\lambda_2(G \text{ⓩ} H) < 1$ whenever $\lambda_2(G) < 1$ and $\lambda_2(H) < 1$. However, we will be dealing with graphs that already have sufficiently large spectral gap that this result will be enough.

Notice that the zig-zag product keeps the degree low while not decreasing the spectral gap by too much. In our final construction, we will need a part of the construction to increase the spectral gap, and it won't matter if that part of the construction increases the degree. We have already seen in a previous lecture that graph squaring has this property, and that is what we will use: an iteration of both zig-zag products and graph squaring.