CS 880: Advanced Complexity Theory	3/3/2008
Lecture 16: Noise Sensitivity of Majority	
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In the first part of this lecture, we analyze the noise sensitivity of the Majority function in more detail than last time. In the second part, we discuss the Majority is Stablest theorem, which says that among balanced functions for which each of the individual variables has low influence, the Majority function has the lowest noise sensitivity up to some small error. We also see some applications of this theorem in social choice theory.

1 Noise Sensitivity of Majority

In this section we derive a good approximation for the noise sensitivity of the Majority function. Let MAJ_n denote Majority on n variables. We do not care about the outputs of MAJ_n on inputs with an equal number of -1's and 1's as the effect will be absorbed by an error term. In our analysis, we replace some distributions in the original problem by normal distributions. The reduced problem turns out to have a nice geometric structure and can be solved more easily. All we lose in the reduction is a small additive error term. We start by writing the noise sensitivity of Majority in the following form.

$$NS_\epsilon(MAJ_n) = \Pr_{\substack{x \\ y \sim_\epsilon x}} \left[\mathrm{sgn}\left(\sum_{i=1}^n x_i\right) \neq \mathrm{sgn}\left(\sum_{i=1}^n y_i\right)\right] \quad (1)$$

$$= \Pr_{\substack{x \\ y \sim_\epsilon x}} \left[\mathrm{sgn}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i\right) \neq \mathrm{sgn}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n y_i\right)\right] \quad (2)$$

Both $\frac{1}{\sqrt{n}}\sum_i x_i$ and $\frac{1}{\sqrt{n}}\sum_i y_i$ converge to normal distributions individually. To see this, we observe that $\mathrm{E}[\frac{1}{\sqrt{n}}\sum_i x_i] = 0$ and the variance is

$$\mathrm{E}\left[\left(\frac{1}{\sqrt{n}}\sum_i x_i\right)^2\right] = \frac{1}{n}\mathrm{E}\left[\sum_{i,j} x_i x_j\right] = \frac{1}{n}\mathrm{E}\left[\sum_i x_i^2\right] = 1.$$

The second equality follows from the independence between x_i and x_j for $i \neq j$, and the fact that $\mathrm{E}[x_i] = 0$. The last equality follows from $|x_i| = 1$.

However, the two distributions are not independent. If we compute the correlation between them, we get

$$\mathrm{E}\left[\left(\frac{1}{\sqrt{n}}\sum_i x_i\right)\left(\frac{1}{\sqrt{n}}\sum_j y_j\right)\right] = \mathrm{E}\left[\frac{1}{n}\sum_i x_i \sum_j y_j\right] = \frac{1}{n}\sum_{i=1}^n \mathrm{E}[x_i y_i] = \epsilon\cdot(-1) + (1-\epsilon)\cdot 1 = 1-2\epsilon.$$

The second equality holds because the cross terms $(\mathrm{E}[x_i y_j], i \neq j)$ evaluate to zero. The fourth equality follows from the definition of y: $y_i \neq x_i$ independently with probability ϵ.

Instead of analyzing the original distributions, we analyze two correlated normal distributions X and Y, and claim that the probability of X and Y having different signs equals the probability of the original distributions having different signs up to some small error. We want the original distributions and the normal distributions to behave similarly. In particular, X and Y should have a correlation of $1-2\epsilon$. To simplify calculation, we can write Y as a linear combination of two normal distributions. Formally, let X and Z be independent normal random variables. Let $Y = (1-2\epsilon)X + \sqrt{1-(1-2\epsilon)^2}Z = (1-2\epsilon)X + 2\sqrt{\epsilon(1-\epsilon)}Z$. It can be verified that Y is also normal and the correlation between X and Y is $1-2\epsilon$. We will see that $\Pr[\mathrm{sgn}(X) \neq \mathrm{sgn}(Y)]$ approximates the quantity in (2). As n tends to infinity, the error becomes negligibly small. We will state the error term without proof at the end.

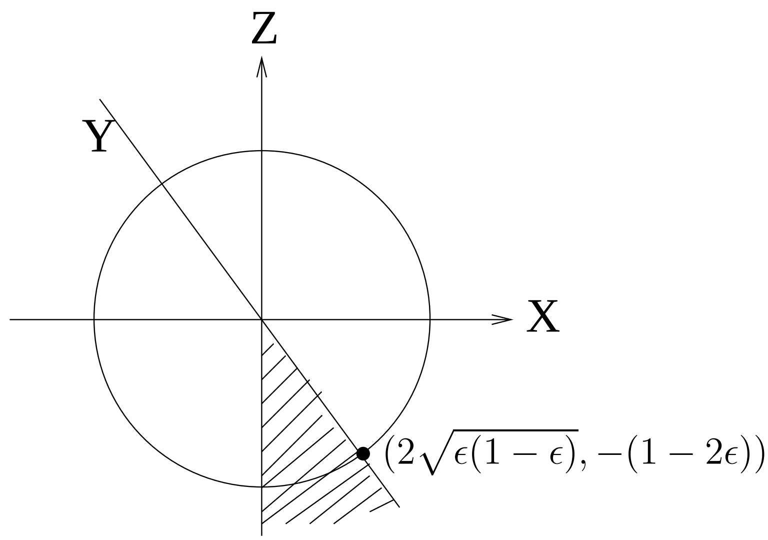

Figure 1: Y is a linear function of X and Z.

Let us compute the probability of X and Y having different signs.

$$\begin{aligned} \Pr[\mathrm{sgn}(X) \neq \mathrm{sgn}(Y)] &= \Pr[X \geq 0 \text{ and } Y < 0] + \Pr[X < 0 \text{ and } Y \geq 0] \quad (3) \\ &= 2\Pr[X \geq 0 \text{ and } Y < 0]. \quad (4) \end{aligned}$$

The last inequality holds by symmetry. We can find out $\Pr[X \geq 0 \text{ and } Y < 0]$ geometrically. Recall that Y is a linear function of X and Z. Figure 1 shows the linear relationship between X, Z and Y. The shaded region in Figure 1 represents the inequalities $X \geq 0$ and $Y < 0$. So $\Pr[X \geq 0 \text{ and } Y < 0]$ is equal to the measure of the shaded region under the joint distribution of X and Z. Since X and Z are independent normal distributions, their joint distribution is symmetric around the origin. Therefore, the measure of the shaded region is $\frac{\theta}{2\pi}$, where $\cos\theta = 1-2\epsilon$. With this we can determine the quantity in Equation (3):

$$\Pr[\mathrm{sgn}(X) \neq \mathrm{sgn}(Y)] = 2 \cdot \frac{\theta}{2\pi} = \frac{\theta}{\pi} = \frac{\arccos(1-2\epsilon)}{\pi}. \quad (5)$$

Also, by writing $\cos\theta = 1 - 2\sin^2(\theta/2) = 1-2\epsilon$, we have

$$\Pr[\mathrm{sgn}(X) \neq \mathrm{sgn}(Y)] = \frac{2}{\pi}\arcsin(\sqrt{\epsilon}). \quad (6)$$

From (5) we get the following theorem. Since the original distributions only converges to normal, we need an error term, which we state without proof.

Theorem 1. *Let MAJ_n denote the Majority function in n variables. (The output can be either way if there is an equal number of 1's and -1's.) Then,*

$$NS_\epsilon(MAJ_n) = \frac{\arccos(1-2\epsilon)}{\pi} \pm O\left(\frac{1}{\sqrt{n\epsilon}}\right).$$

If ϵ is small, then $\arcsin \epsilon \approx \epsilon$. Theorem 1 and (6) give the following corollary.

Corollary 1.

$$NS_\epsilon(MAJ_n) \sim \frac{2}{\pi}\sqrt{\epsilon}.$$

2 Majority is Stablest

In this section we cover the Majority is Stablest theorem, which says that the Majority function is stablest in the sense that it has the lowest noise sensitivity among balanced Boolean functions for which individual variables have low influences. A stronger version of this theorem says that this also holds if we consider all balanced functions from $\{-1,1\}^n$ to $[-1,1]$. We will need this stronger result later so we state it here. Before doing so, we need to generalize the definitions of balancedness and influence to functions $f : \{-1,1\}^n \to [-1,1]$.

- f is balanced if $E_x[f(x)] = 0$.
- Define the influence of the i^{th} variable to be

  $$I_i(f) = \mathrm{E}_x\left[\sigma^2(f|_{x,i})\right], \tag{7}$$

  where $f|_{x,i}$ is the restriction of f obtained by fixing all variables except the i^{th} according to x. When f is Boolean, this new definition coincides with our old definition $I_i(f) = Pr[f(x) \neq f(x^{(i)})]$. It can be shown that (7) equals $||D_i f||^2$ (and $\mathrm{E}[(D_i f)^2]$ in the Boolean case). By Parseval's equality, this is equal to $\sum_{S \ni i}(\hat{f}(S))^2$.

We state the following theorem without proof.

Theorem 2. *Let f be a function from $\{-1,1\}^n$ to $[-1,1]$ with $\mathrm{E}[f] = 0$ such that $I_i(f) \leq \tau$ for all $i \in [n]$. Then, for any $0 < \epsilon < \frac{1}{2}$,*

$$NS_\epsilon(f) \geq \frac{1}{\pi}\arccos(1-2\epsilon) - O\left(\frac{1}{\epsilon} \cdot \frac{\log\log\frac{1}{\tau}}{\log\frac{1}{\tau}}\right).$$

If we compare Theorem 2 and Theorem 1, we see that Majority is indeed the stablest up to a small error term. We can strengthen the theorem a little bit by bounding the Fourier weight of subsets of size at most $\log(1/\tau)$.

Theorem 3 (Majority is Stablest Theorem)**.** *Let f be a function from $\{-1,1\}^n$ to $[-1,1]$ with* $\mathrm{E}[f]=0$ *such that for all $i \in [n]$,*

$$\sum_{\substack{S \ni i \\ |S| \le \log(1/\tau)}} \left(\hat{f}(S)\right)^2 \le \tau.$$

Then, for any $0 < \epsilon < \frac{1}{2}$,

$$NS_\epsilon(f) \ge \frac{1}{\pi}\arccos(1-2\epsilon) - O\left(\frac{1}{\epsilon} \cdot \frac{\log\log\frac{1}{\tau}}{\log\frac{1}{\tau}}\right).$$

Remark: The proof of Theorem 3 is similar to what we did in the proof of Theorem 1. We approximate the original distributions with normal distributions and argue that the error is small. Then the problem becomes one with nice geometric structure which can be solved exactly.

2.1 Reverse Majority is Stablest Theorem

In general, the probability of flipping a variable could be any $0 < \epsilon < 1$. However, in Theorem 3 we require ϵ to be less than $\frac{1}{2}$. What happens if we allow ϵ to lie in the other half, i.e., $\frac{1}{2} < \epsilon < 1$? It turns out that the inequality turns around. Formally, we have the following theorem. Note that we no longer require the function to be balanced.

Theorem 4 (Reverse Majority is Stablest Theorem)**.** *Let f be a function from $\{-1,1\}^n$ to $[-1,1]$ such that for all $i \in [n]$,*

$$\sum_{\substack{S \ni i \\ |S| \le \log(1/\tau)}} \left(\hat{f}(S)\right)^2 \le \tau.$$

Then, for any $\frac{1}{2} < \epsilon < 1$,

$$NS_\epsilon(f) \le \frac{1}{\pi}\arccos(1-2\epsilon) + O\left(\frac{1}{1-\epsilon} \cdot \frac{\log\log\frac{1}{\tau}}{\log\frac{1}{\tau}}\right).$$

Proof. Let $\frac{1}{2} < \epsilon < 1$. We first assume that f is odd, i.e. $f(-x) = -f(x)$. In that case, we argued last lecture that

$$NS_{1-\epsilon}(f) = 1 - NS_\epsilon(f). \tag{8}$$

Also note that

$$\begin{aligned} \frac{1}{\pi}\arccos(1-2(1-\epsilon)) &= \frac{1}{\pi}\arccos(-(1-2\epsilon)) && (9)\\ &= 1 - \frac{1}{\pi}\arccos(1-2\epsilon). && (10) \end{aligned}$$

Since $0 < 1-\epsilon < \frac{1}{2}$, and every odd function is balanced, we can apply Theorem 3. We get

$$NS_{1-\epsilon}(f) \geq \frac{1}{\pi}\arccos(1-2(1-\epsilon)) - O\left(\frac{1}{1-\epsilon}\cdot\frac{\log\log\frac{1}{\tau}}{\log\frac{1}{\tau}}\right) \quad (11)$$

$$1 - NS_{\epsilon}(f) \geq 1 - \frac{1}{\pi}\arccos(1-2\epsilon) - O\left(\frac{1}{1-\epsilon}\cdot\frac{\log\log\frac{1}{\tau}}{\log\frac{1}{\tau}}\right) \quad (12)$$

$$NS_{\epsilon}(f) \leq \frac{1}{\pi}\arccos(1-2\epsilon) + O\left(\frac{1}{1-\epsilon}\cdot\frac{\log\log\frac{1}{\tau}}{\log\frac{1}{\tau}}\right). \quad (13)$$

In the above, (12) follows from (8) and (10).

We have proved the theorem for odd f. For general f, let f_{odd} be the odd part of f, i.e., $f_{odd} = \sum_{|S|\ odd} \hat{f}(S)\chi_S$. We claim that $NS_\epsilon(f) \leq NS_\epsilon(f_{odd})$. To prove this, recall that $NS_\epsilon(f) = \frac{1}{2} - \frac{1}{2}\sum_S (1-2\epsilon)^{|S|}(\hat{f}(S))^2$. If we drop all the terms with even $|S|$, we get $NS_\epsilon(f_{odd})$. For these terms, $(1-2\epsilon)^{|S|}$ is positive, thus proving our claim. We still need to argue that $f_{odd}(x) \in [-1,1]$ and that f_{odd} satisfies the low influence condition. The former follows because $f_{odd}(x) = (f(x) - f(-x))/2$, and the latter because for each $S \subseteq [n]$, $\widehat{f_{odd}}(S) \leq \hat{f}(S)$. This concludes our proof. □

2.2 Applications in Social Choice Theory

In this section, we look at some applications of the Majority is Stablest theorem in social choice theory.

2.2.1 Two Candidate Elections

Recall that an election with two candidates and n voters can be modeled as a function $f : \{-1,1\}^n \to \{-1,1\}$ where the candidates are labeled -1 and 1 respectively, $x \in \{-1,1\}^n$ indicates the preferences of voters, and $f(x)$ gives the result of the election given preferences x. We also assume an "impartial culture" (IC), i.e., the preferences of voters are independently and uniformly distributed. The Majority is Stablest theorem implies that among all fair (balanced) election systems in which individual voters have small influences, the Majority election system is the stablest against random noise in the input, provided the noise is not so great that input bits get flipped by default, and up to low order terms.

2.2.2 Three Candidate Elections

In lecture 12 we looked at elections with three or more candidates. We discussed the Condorcet method in which each voter ranks all of the candidates. An aggregate function f is applied to each pair of candidates to determine the winner. The hope is that this gives a ranking of the candidates. Condorcet's Paradox shows that this procedure fails if f is the majority function even when there are only three candidates and three voters. Arrow's theorem tells us that the only functions that always maintain rationality are dictators and anti-dictators. However, dictators and anti-dictators are not very useful in reality – Majority would be a more common choice. The reason is that when seeking a good voting scheme, we want the influences of individual voters to be small. Given this restriction, Majority turns out to be the best choice, in the sense that it has the highest probability of maintaining rationality under the IC assumption. Let f be any Boolean function. Define the rationality of f as

$$\begin{aligned} RATIONALITY(f) &= \Pr[\text{aggregate graph has no cycle}] \\ &= \Pr_{\substack{x,y,z \\ s.t.(\forall i)\neg(x_i=y_i=z_i)}}[\neg(f(x)=f(y)=f(z))]. \end{aligned}$$

Using Problem 2 of Homework 1, we get

$$RATIONALITY(f) = \frac{3}{4} + \frac{1}{4}\sum_{S\subseteq[n]}\left(\frac{-1}{3}\right)^{|S|-1}(\hat{f}(S))^3 = \frac{3}{2}NS_{2/3}(f).$$

If we require that f has small individual influences, then by the Reverse Majority is Stablest theorem, the above quantity is maximized (up to small error) when f is Majority. Formally, we have the following theorem.

Theorem 5. *If f is a Boolean function satisfying $I_i(f) \leq \tau$ for all $i \in [n]$, then*

$$RATIONALITY(f) \leq RATIONALITY(MAJ_n) + O\left(\frac{\log\log\frac{1}{\tau}}{\log\frac{1}{\tau}}\right) + O\left(\frac{1}{\sqrt{n}}\right).$$

3 Next Time

Next time we will see hardness of approximation results based on harmonic analysis. We will discuss the maximum satisfiability problem, the maximum cut problem and the vertex cover problem. The Reverse Majority is Stablest Theorem will play a role in the hardness result for maximum cut.