CS/Math 240: Intro to Discrete Math 5/11/2011

Final Exam

Instructor: Dieter van Melkebeek

Guidelines:

- **Do NOT turn this page until you have received the signal to start. In the mean time read the instructions below carefully.**
- Fill in your name, student ID, and circle your discussion section below.
- This booklet consists of 7 sheets of paper, containing guidelines and 6 questions. When you receive the signal to start, please make sure that your copy of the test is complete.
- Do not separate the pages of this booklet.
- Answer each question directly on this booklet, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and indicate clearly the part of your work that should be marked.
- In your answers, you may use without proof any result or theorem covered in lectures, lecture notes, discussion sections, or homework, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.
- Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do.
- Good luck!

Identifying Information:

Name:

Student ID:

Section:

301	302	303	304	305
Nilay	Nilay	Roger	Roger	Dalibor
M 9:55-10:45	W 9:55-10:45	M 12:05-12:55	W 12:05-12:55	W 1:20-2:10

For TA use only:

Score: / 25

Question 1: [5 points]

For two relations R_1 and R_2 on a domain D, we define the composition of R_2 after R_1 as

$$R_2 \circ R_1 = \{(x, z) \in D \times D \mid (\exists y)\, (x, y) \in R_1 \wedge (y, z) \in R_2\}.$$

(a) Let R denote the relation on the domain of all people such that $(x, y) \in R$ iff x is the biological mother of y. What is the meaning of the relation $R \circ R^{-1}$? Is it an equivalence relation? If so, what are the equivalence classes?

(b) Which of the following statements is true of every relation R on a domain D? Either give a proof or a counterexample.

(i) $R \circ R^{-1}$ is reflexive.

(ii) $R \circ R^{-1}$ is symmetric.

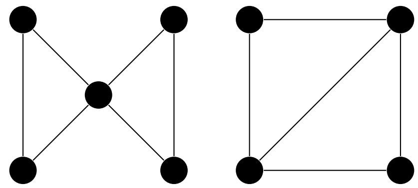
(iii) $R \circ R^{-1}$ is transitive.

Question 2: [3 points]

Show that in every graph every vertex of odd degree has a path to some other vertex of odd degree.

Question 3: [3 points]

Consider the following graph $G = (V, E)$:



The graph G is clearly planar, but the following "argument" concludes that it is not.

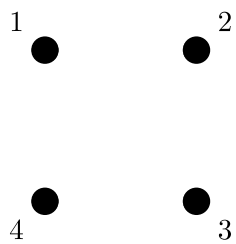
> Suppose that G has a planar embedding with faces F. By Euler's formula we have that $|V| - |E| + |F| = 2$. Define the set $B = \{(e, f) \in E \times F \mid e \text{ is on the border of } f\}$. Since every simple cycle in G is of length 3, every face has 3 edges on its border, so $|B| = 3|F|$. On the other hand, every edge borders 2 faces, so $|B| = 2|E|$. Therefore, $|F| = \frac{2}{3}|E|$, which together with Euler's formula yields that $|V| - \frac{1}{3}|E| = 2$, so $|E| > 3|V|$. However, the latter inequality contradicts the fact that $|V| = 9$ and $|E| = 11$. We conclude that G is not planar.

Find *all* logical errors in the above argument.

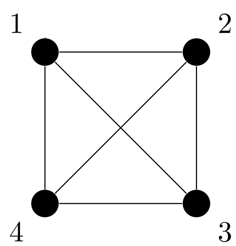
Question 4: [4 points]

Consider the following four graphs with labeled vertices, and determine for each of them in how many ways you can color its vertices with k or fewer colors ($k \in \mathbb{N}$) such that no two adjacent vertices have the same color.

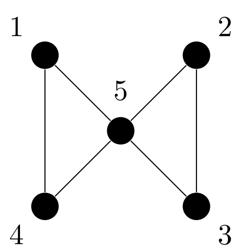
(a)



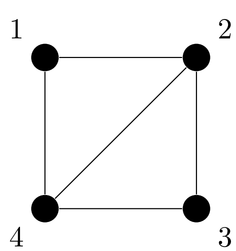
(b)



(c)



(d)



Question 5: [5 points]

Consider the language L of all binary strings that contain an even number of 0's or exactly two 1's.

(a) Construct a nondeterministic finite automaton with 6 states that accepts L.

(b) Construct a deterministic finite automation with 8 states that accepts L.

(c) Construct a regular expression that defines L.

Question 6: [5 points]

Show that for every integer $n \geq 1$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^{n-1}$$

in two different ways.

(a) Using a combinatorial argument.

(b) Using the binomial theorem.