CS/Math 240: Intro to Discrete Math 05/07/2011

Practice Problems for the Final Exam

Instructor: Dieter van Melkebeek

- This problem set is intended to help you prepare for the final. The actual exam will be shorter, and the questions will be spaced so you can answer them on the sheets you are given. The actual exam will also spell out more guidelines.
- The solutions to the sample exam will be discussed during the review sessions of 5/9 and 5/10.

Program Analysis

1. Recall the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ from class. Using this identity, we can make a recursive program for calculating binomials as follows:

   Binomial(n, k)
   (1) **if** $n = k$ **then return** 1
   (2) **if** $k = 0$ **then return** 1
   (3) **return** Binomial$(n-1, k-1)$ + Binomial$(n-1, k)$

   Determine the number of recursive calls incurred by the Binomial algorithm in terms of n, k.

Relations

2. How many possible relations are there from a n-element set A to itself? How many possible relations are there which are reflexive? How many possible relations are there which are both symmetric and reflexive?

Graph Theory

3. Define a Hamiltonian circuit as a cycle of length n on an n-vertex graph that contains no repeated vertices.

   (a) How many distinct Hamiltonian circuits exist on a fully connected graph of n vertices?

   (b) Prove the following: a bipartite graph with an odd number of vertices has no Hamiltonian circuit.

4. Prove the following: A bipartite graph has a unique bipartition (besides switching the labels on each partition) if and only if it is connected.

Finite Automata

5. Show that for any regular language L, the reverse $L^R = \{x \in \Sigma^* \mid x^R \in L\}$ is also regular, where x^R denotes the string x in reverse. For example, if $\Sigma = \{0, 1\}$ and L_1 denotes the set of strings beginning with 10, then L_1^R denotes the set of strings ending with 01.

6. Show that for any regular language L, the complement $\bar{L} = \{x \in \Sigma^* \mid x \notin L\}$ is also regular. For example, if $\Sigma = \{0, 1\}$ and L_1 denotes the set of strings with even parity, then $\bar{L_1}$ denotes the set of strings with odd parity. Further, show that for any two regular languages L_1 and L_2 over the same alphabet, the intersection $L_1 \cap L_2$ is also regular.

Counting

7. How many ways are there to place two rooks on a chessboard so that they are threatening each other (either on the same row or column)?

8. Show that $k\binom{n}{k} = n\binom{n-1}{k-1}$ using a combinatorial proof.

9. Consider a group of ten people, each between 1 and 60 years old (inclusive). Prove that we can pick two disjoint sets of people in the group such that the sum of ages in each group is equal. These sets do not need to form a partition. (For example, if we had a group of people with ages $1, 2, \ldots, 10$, the sets $(1, 10)$ and $(2, 9)$ would have the same sum of ages).

   Extra: Can we make the same claim with a group of less people?