CS/Math 240: Intro to Discrete Math 2/3/2011

Homework 2

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Guidelines:

- This assignment covers sets and proof paradigms.
- The general guidelines from the first assignment still apply, and will do so for the rest of the semester. You can state results from class without reproving them. Argue all other claims you make.
- The assignment is due on 2/10 at the beginning of class.
- Good luck!

Questions:

1. Let A, B, and C denote arbitrary finite sets.
   (a) Show that $|A \cup B| = |A| + |B| - |A \cap B|$.
   (b) Derive a similar equation for three sets. The left-hand side should be $|A \cup B \cup C|$ and the only set operator allowed on the right-hand side is intersection.
2. Consider the domain of the positive integers. Recall that $\gcd(a, b)$ denotes the greatest common divisor of a and b, and $\mathrm{lcm}(a, b)$ their least common multiple. Let M_a denote the set of multiples of a, i.e., $M_a = \{a, 2a, 3a, \ldots\}$.

   Prove or disprove the following propositions.
   (a) $(\forall a, b)\, M_a \cap M_b = M_{\mathrm{lcm}(a,b)}$
   (b) $(\forall a, b)\, M_a \cup M_b = M_{\gcd(a,b)}$
3. Which of the following statements about countable and uncountable sets are true? Prove your assertions.
   (a) There exist uncountable sets A and B such that $A - B$ and $A \cap B$ are countable.
   (b) There exist uncountable sets A and B such that $A - B$ is uncountable and $A \cap B$ is countable.
   (c) There exist uncountable sets A and B such that $A - B$ is countable and $A \cap B$ is uncountable.
   (d) There exist uncountable sets A and B such that $A - B$ and $A \cap B$ are uncountable.
   (e) For every mapping S of a countable set A to countable sets, the union $\cup_{x \in A} S(x)$ is countable.
   (f) For every mapping S of a set A to countable sets, the union $\cup_{x \in A} S(x)$ is countable.

4. Call a real number x root-rational if there exist positive integers a and b such that $\sqrt{a/b} = x$.

   (a) Show that $\sqrt[3]{2}$ is not root-rational.

   (b) Describe exactly for which natural numbers n, $\sqrt[3]{n}$ is root-rational.

5. In class we showed that every group of 6 people has the following property: The group includes a subgroup of 3 people that are mutual acquaintances or a subgroup of 3 people that are mutual strangers.

   (a) Show that the above property does not hold in general when we replace 6 by 5.

   (b) Find all possible relationships between 5 people for which the property does not hold.

Extra Credit:

Generalize problem 1 to every finite number of sets.