CS/Math 240: Intro to Discrete Math 2/10/2011

Homework 3

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Guidelines:

This assignment covers induction. It is due on 2/17 at the beginning of class. Good luck!

Questions:

1. Show that for every $n \in \mathbb{N}$,
$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

2. Show that for any finite number of finite sets $A_1, A_2, \ldots, A_n$,
$$|\cup_{i=1}^{n} A_i| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} |\cap_{i \in I} A_i|.$$
Here $[n]$ is a shorthand for $\{1, 2, \ldots, n\}$. The sum ranges over all nonempty subsets I of $[n]$. Think of I as an index sets that selects which of the sets A_i to take the intersection of.

3. Find the flaw in the following *bogus* proof by strong induction that for all natural numbers n, $a^n = 1$ whenever a is a nonzero real number.

   Let $P(n)$ denote the statement that for all real numbers $a \neq 0$, $a^n = 1$.

   Base Case: $P(0)$ is equivalent to $a^0 = 1$, which is true by definition of a^0.

   Inductive Step: Let a by any nonzero real number. By the induction hypothesis, $a^k = 1$ for all $k \in \mathbb{N}$ such that $k \leq n$. But then
$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$
   which implies that $P(n+1)$ holds.

   It follows by induction that $P(n)$ holds for all $n \in \mathbb{N}$.

4. There are two types of creatures on Mars: A-lings and B-lings. Furthermore, every creature belongs to a particular generation. The creatures in each generation reproduce according to certain rules and then die off. The subsequent generation consists entirely of their offspring.

   The creatures of Mars pair with a mate in order to reproduce. First, as many A-B pairs as possible are formed. The remaining creatures form A-A pairs or B-B pairs, depending on whether there is an excess of A-lings or of B-lings. If there are an odd number of creatures, then one in the majority species dies without reproducing. The number and type of offspring is determined by the types of the parents

   - If both parents are A-lings, then they have three A-ling offspring.
   - If both parents are B-lings, then they have two B-ling offspring and one A-ling offspring.

- If there is one parent of each type, then they have one offspring of each type.

There are 200 A-lings and 800 B-lings in the first generation.

Show that the number of A-lings will always be at most twice the number of B-lings.

5. Consider the following two-person game. Some positive number n of sticks are placed on the ground. The players take turns removing one, two, or three sticks. The player to remove the last one loses.

   (a) Determine for which values of n the first player has a winning strategy, i.e., a strategy that guarantees a win no matter what strategy the other player uses.

   (b) Formalize and prove the following statement: In any two-person game that ends after a finite number of steps and in which each player sees the moves of the other player, either the first player has a winning strategy, or the second one has.

Extra Credit:

On Venus every individual lives forever, except possibly those who have a mark on their forehead. Such an indivual dies the morning after finding out about the mark (if ever). Every day at noon all people on Venus gather together to see each other but no one is ever allowed to discuss the marks. There are also no mirrors on Venus. One day an Earthling passes by Venus. During a speech at the daily meeting, the Earthling ignorantly proclaims: "I see someone who has a mark on the forehead."

Suppose there are n people on Venus with a mark on their forehead, and that all are master logicians. What happens?