CS/Math 240: Intro to Discrete Math 3/3/2011

Homework 5

Instructor: Dieter van Melkebeek

Guidelines:

This assignment covers program analysis, recursion, and recurrences. It is due on 3/10 at the beginning of class. Good luck!

Questions:

1. Consider the following program specification:

   Input: An integer $n \geq 0$ and an array $A[0..n-1]$ of integers.

   Output: The array A sorted from smallest to largest.

   Consider the following implementation, known as bubblesort – the algorithm consists of a number of phases that send a bubble from left to right that swaps two neighboring elements if they are out of order.

   BubbleSort(n, A)
   (1) $m \leftarrow n$
   (2) **while** $m > 1$
   (3) $\ell \leftarrow 0$
   (4) $i \leftarrow 0$
   (5)   **while** $i < m - 1$
   (6)   **if** $A[i] > A[i+1]$
   (7)   swap $A[i]$ and $A[i+1]$
   (8) $\ell \leftarrow i + 1$
   (9) $i \leftarrow i + 1$
   (10) $m \leftarrow \ell$

   (a) State and prove adequate loop invariants for the inner and outer while loops.
   Hint: What can you say about the position of the largest element in A after one iteration of the outer while loop? What can you say about $A[m..(n-1)]$?

   (b) Use the invariants to prove partial correctness and termination.

   (c) Determine exactly the maximum number of times the test in line (5) is executed as a function of n.

2. Consider the following program:

```
FastExp(a, b)
(1)   if b = 0 then return 1
(2)   if b is even then return Square(FastExp(a, b/2)
(3)                else return a· FastExp(a, b − 1))
```

The program makes use of the subroutine Square(x), which simply computes $x \cdot x$ and returns that value.

(a) Show that FastExp correctly implements the following specification:

Input: Integers a, b with $a \neq 0$ and $b \geq 0$.

Output: a^b.

As usual, this involves establishing partial correctness and termination.

(b) How many recursive calls are made when FastExp is called with $b = 2^k$ for some integer k? How often will two elements be multiplied?

(c) How do the answers to the previous question change when you replace line (2) in the code for FastExp by the following: "return FastExp($a, b/2$) · FastExp($a, b/2$)"?

3. Recall that the Fibonacci sequence is defined as the unique solution to the following two sets of conditions.

Initial conditions: $X_1 = X_2 = 1$.

Recurrence condition: $(\forall n \geq 3)\, X_n = X_{n-1} + X_{n-2}$.

In class we stated the following closed form for the Fibonacci sequence (as a suggested exercise on induction):

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}, \tag{1}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.

The expression (1) came out of the blue. In this problem, we'll derive the expression and give an alternate proof.

(a) Determine all reals c such that $X_n = c^n$ satisfies the recurrence condition (but not necessarily the intial conditions).

(b) Show that if $X_n = A_n$ and $X_n = B_n$ satisfy the recurrence condition, then so does $X_n = \alpha \cdot A_n + \beta \cdot B_n$ for every choice of reals α and β.

(c) Use parts (a) and (b) to show that the Fibonacci sequence satisfies (1).

4. Solve the following recurrence: $A_0 = 3$, $A_1 = 7$ and $A_n = 3A_{n-1} - 2A_{n-2}$ for all integers $n \geq 2$.