**CS/Math 240: Intro to Discrete Math** 3/31/2011

# Homework 7

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**Guidelines:**

This assignment covers relations, functions, and digraphs. It is due on 4/7 at the beginning of class. Good luck!

**Questions:**

1. Let $f$ be a function from $A$ to $B$. For any subset $X \subseteq A$, we denote by $f(X)$ the set $\{f(x) \,|\, x \in X\}$.

   Show that $f$ is one-to-one iff $(\forall X, Y \subseteq A)\, f(X \cap Y) = f(X) \cap f(Y)$.

2. Consider the following relations on the domain of all subsets of some finite set $D$ with $|D| \geq 2$.

   (a) $R_1 = \{(x, y) \,|\, x \cap y = \emptyset\}$

   (b) $R_2 = \{(x, y) \,|\, x \cup y = \emptyset\}$

   (c) $R_3 = \{(x, y) \,|\, |x| \leq |y|\}$

   (d) $R_4 = \{(x, y) \,|\, x \cup y = x \cap y\}$

   (e) $R_5 = \{(x, y) \,|\, \text{there exists a total function from } x \text{ to } y \text{ that is a bijection}\}$

   Determine for each relation (i) whether it is an equivalence relation, and (ii) whether it is an order relation. In the case of an equivalence relation, determine the equivalence classes. In the case of an order relation, determine whether it is total.

3. Consider the domain $D$ of all finite sequences built from 0s and 1s, including the empty sequence. For any two such sequences $x$ and $y$ we denote by $xy$ the sequence consisting of $x$ followed by $y$.

   (a) Show that for every subset $L$ of $D$, the following is an equivalence relation on $D$:
   $R_L = \{(x, y) \,|\, (\forall z)\, xz \in L \Leftrightarrow yz \in L\}$.

   (b) What are the equivalence classes of $R_L$ when

      i. $L$ consists of all strings with an even number of 1s,

      ii. $L$ consists of all strings such that the number of 1s is a multiple of $k$, and

      iii. $L$ consists of all strings with exactly $k$ 1s.

   In the last two cases $k$ denotes an arbitrary natural number.

4. Given an order relation $R$ on a domain $D$ and two elements $x, y \in D$, we say that $z \in D$ is a supremum of $x$ and $y$ if (i) $x \, R \, z$, (ii) $y \, R \, z$, and (iii) $(\forall z') \, (xRz' \wedge yRz' \Rightarrow zRz')$. We say that $z \in D$ is an infimum of $x$ and $y$ for $R$ if $z$ is a supremum of $x$ and $y$ for $R^{-1}$.

   Determine for each of the following order relations whether a supremum and infimum exists for all $x, y \in D$, and if so, what they are.

   (a) $\leq$ on the set of integers,

   (b) $<$ on the set of integers,

   (c) $|$ on the set of positive integers, and

   (d) $\subseteq$ on the set of subsets of the integers.

5. Consider the following game.

   - There are $n$ people playing, one of whom leads the game. They are playing on a playing field with no obstacles. Everyone carries one water balloon.
   - Everyone walks around the playing field until the game leader yells "stop". Assume that the leader will yell stop only when everyone is in a position where they have a unique closest neighbor.
   - Next, the leader yells "throw" and everyone throws their water balloon at their nearest neighbor (there is exactly one, by the assumption in the previous point).
   - A survivor is anyone who is still dry at the end of the game (assuming everyone has perfect aim and the water balloons are designed so that they get only their intended target wet).

   Prove that for all odd natural numbers $n$, there will always be at least one survivor.