CS/Math 240: Intro to Discrete Math 4/14/2011

Homework 9

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Guidelines:

This assignment covers trees, graph coloring, and finite state machines. It is due on 4/21 at the beginning of class. Good luck!

Questions:

1. Show that a connected graph $G = (V, E)$ with $|V| = |E| + 1$ is a tree.

2. Consider the following program, which takes as input a tree $T = (\{1, 2, 3, \ldots, n\}, E)$ on $n \geq 2$ vertices, and outputs an array $A[1..n-2]$.

   TreeRepresentation(T, n)
   (1) $G \leftarrow T$
   (2) **for** $i = 1$ **to** $n - 2$
   (3) $\quad \ell \leftarrow$ leaf in G with the smallest label
   (4) $\quad A[i] \leftarrow$ unique neighbor of ℓ in G
   (5) $\quad G \leftarrow G \setminus \ell$
   (6) **return** $A[1..n-2]$

   (a) Construct an algorithm that takes as input an integer $n \geq 2$ and an array A of length $n - 2$ with entries in $V \doteq \{1, 2, 3, \ldots, n\}$, and produces a tree $T = (V, E)$ such that TreeRepresentation$(T, n) = A$.

   (b) Is the construction always possible? Is the answer uniquely determined? What can you conclude about the number of distinct trees on the vertex set $V = \{1, 2, 3, \ldots, n\}$?

3. For a given graph G, we define the chromatic function $P_G : \mathbb{N} \to \mathbb{N}$ as the function that maps a natural number x to the number of valid colorings of G with at most x colors.

   (a) Determine P_T for all trees T on n vertices.

   (b) Let $e = (u, v)$ be an edge in G, and let G'_e denote the result of contracting e in G. Argue that $P_{G'_e}(x)$ equals the number of valid colorings of $G \setminus e$ with at most x colors in which u and v get assigned the same color.

   (c) Show that P_G is a polynomial. *Hint:* Use induction.

   (d) Express the following in terms of characteristics of G: (i) the degree of P_G, (ii) the coefficient of the highest-order term, (iii) the coefficient of the term of degree one less.

4. For any nonnegative integer k, define D_k as the language of all ternary representations of multiples of k over the alphabet $\{0, 1, 2\}$. Leading zeroes are not allowed.

   Give finite automata accepting the following languages:

$$\begin{aligned} &D_0 \\ &D_1 \\ &D_2 \\ L_1 &= (D_2 \cup \{\epsilon\}) \setminus \{0\} \\ L_2 &= L_1 D_2 \\ L_3 &= L_1^* \{0\} \\ L_4 &= (D_2 \setminus \{0\})\{xx \mid x \in \{0,1,2\}^*\} \cup \{0\} \end{aligned}$$

5. In a monastery each monk is given a bowl with 15 red beads and 12 green beads. Each day at noon, a monk must do one of two things:

   - Exchange: If he has at least 3 red beads in his bowl, then he may exchange 3 red beads for 2 green beads.
   - Swap: He may replace each green bead in his bowl with a red bead and replace each red bead in his bowl with a green bead. That is, if he starts with r red beads and g green beads, then after he performs this operation, he has g red beads and r green beads.

   A monk may leave the monastery only when he has exactly 5 red beads and 5 green beads in his bowl.

   (a) Model this process using a finite state machine.

   (b) Show that no monk can ever leave the monastery.

   Extra credit: Determine the maximum number of distinct states a monk can be in during his life.