CS/Math 240: Intro to Discrete Math 5/3/2011

Homework 11

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Guidelines:

This assignment covers counting. You do not need to turn it in, but should do the problems. Model solutions will be posted on 5/5.

Questions:

1. Solve the following counting problems.

   (a) How many poker hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

   (b) How many poker hands have two or more kings?

   (c) How many poker hands contain the ace of spades, the ace of clubs, or both?

   (d) For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

   $$\sum_{i=0}^{k} x_i = n$$

   (e) For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

   $$\sum_{i=0}^{k} x_i \leq n$$

   (f) In how many ways can $2n$ students be broken up into n groups of 2?

   (g) How many simple undirected graphs are there with vertex set $\{1, 2, \ldots, n\}$?

2. A derangement is a permutation $(x_1, x_2, \ldots, x_n)$ of the set $\{1, 2, \ldots, n\}$ such that $x_i \neq i$ for all i. For example, $(2, 3, 4, 5, 1)$ is a derangement, but $(2, 1, 3, 5, 4)$ is not because 3 appears in the third position. The objective of this problem is to count derangements.

   (a) Derive a formula for the number of derangements of length n using inclusion-exclusion.

   (b) Show that the fraction of permutations on n elements that are derangements is asymptotically equivalent to $1/e$. *Hint:* Recall that $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$.

3. Prove that among any $n^2 + 1$ points within an $n \times n$ square there must exist two points whose distance is at most $\sqrt{2}$.

4. Give a combinatorial proof of the following identity:

   $$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

Extra:

Here is a magic trick. The magician's assistant goes into the audience with a standard deck of 52 cards while the magician looks away. Five audience members each select one card from the deck. The assistant then gathers up the five cards and holds up four of them so the magician can see them. The magician concentrates for a short time and then correctly names the secret, fifth card!

Explain how the magician and his assistant can perform this trick.