1 Quiz 1 Solutions

1.1 Monday 12:05 - 12:55 Quiz

(1) *Given $L(x, y)$ as the predicate 'x likes y,' translate the following.*

(a) *Nobody who likes Alice likes Bob.*

From the statement, we see that for any person in the universe, it is not possible for them to like Bob, given that they like Alice. This translates to either $\forall x(L(x, Alice) \Rightarrow \neg L(x, Bob))$ or $\neg\exists x(L(x, Alice) \wedge L(x, Bob))$.

(b) $\forall x(L(Bob, x) \Rightarrow \neg\exists y(y \neq Bob \wedge L(y, x)))$

We first examine the consequent of our implication, $\neg\exists y(y \neq Bob \wedge L(y, x))$. This translates as "There is not a person y in the domain who is not Bob and likes x." Combining this with the antecedent gives "If Bob likes x, then there is not a person y in the domain who is not Bob and likes x." Since the statement is surrounded by a universal quantifier, we see that this statement is true for all x in the domain. So "Everyone who Bob likes is not liked by anyone else."

(2) *Given the following statement:*

If the product of two integers xy is even, then x is even or y is even.

(a) *Write the contrapositive of this statement, either in English or as a propositional statement.*

We can separate the original statement into three propositions:
P: The product xy is even
Q: x is even
R: y is even

making the whole statement $P \Rightarrow (Q \vee R)$. The contrapositive of a statment $X \Rightarrow Y$ is $\neg Y \Rightarrow \neg X$; applying this to our statement gives $\neg(Q \vee R) \Rightarrow \neg P$. Applying DeMorgan's Law to the statement results in $(\neg Q \wedge \neg R) \Rightarrow \neg P$. Translating back into English, "If x is odd [not even] and y is odd, then the product xy is odd."

(b) *Prove the statement*

It is slightly easier to prove the contrapositive of the original statement; since the two are equivalent, proving the contrapositive is sufficent to prove the original. If x and y are both odd, then they can be written as $2n + 1$ and $2m + 1$ for some integers m and n. So their product can be written as $(2n + 1)(2m + 1) = 4mn + 2m + 2n + 1 =$

$2(2mn+m+n)+1$, which is clearly odd as $2(mn+m+n)$ is an integer. So our proof is complete.

1.2 Wednesday 12:05 - 12:55 Quiz

(1) *Given $L(x,y)$ as the predicate 'x likes y,' translate the following.*

(a) *Only people who like Alice dislike Bob.*

From this statement, we see that for any person in the universe, it is not possible for that person to dislike Bob if they dislike Alice. So this translates into either $\forall x(\neg L(x, Bob) \Rightarrow L(x, Alice))$ or $\neg\exists x(\neg L(x, Bob) \wedge \neg L(x, Alice))$.

(b) $\neg\exists x\forall y(x \neq Bob \wedge (L(Bob, y) \Rightarrow L(x, y)))$

The two quantifiers together read "There does not exist an x, such that for all y in the domain [the following holds]." The encapsulated statement reads "x is not Bob, and if Bob likes y, then x likes y." Since y is universal, this translates to "Everyone Bob likes, x also likes." Finally, applying the negated existential quantifier, we arrive at "There is no person [other than Bob] who likes everyone that Bob likes."

Note that it is still possible for a given x to like a subset of people that Bob likes. As long as there is one person who Bob likes that the specific x does not, the universal quantifier fails [and thus the negated existential holds for this x].

(2) *Given the following statement:*

Given three integers x, y, and z, if $(x+y)$ and $(y+z)$ are both even integers, $(x+z)$ is even.

(a) *Prove or disprove the original statement.*

We will make a proof by cases, with the first case being "x is odd" and the second "x is even". In the first case, for $(x+y)$ to be even with x odd, y must be odd; then, for $(y+z)$ to be even, z must be odd as well. So $(x+z)$ is (odd + odd), which must be even.

In the second case, for $(x+y)$ to be even with x even, y must be even; then, for $(y+z)$ to be even, z must be even as well. So $(x+z)$ is (even + even), which must be even. We see that our cases cover all possibilities (x can never be neither odd nor even), so our proof is complete.

(b) *Write the converse of this statement, either in English or as a propositional statement. Prove or disprove this statement.*

We can separate our original statement into three propositions:
P: (x + y) is even
Q: (y + z) is even
R: (x + z) is even

making the whole statement $(P \wedge Q) \Rightarrow R$. The converse of a statement $X \Rightarrow Y$ is $Y \Rightarrow X$; applying this to our statement gives $R \Rightarrow (P \wedge Q)$, or in English "If (x + z) is even, then (x + y) and (y + z) are both even." This statement is clearly false, as can be seen with the assignment $x = 2, y = 3, z = 4$.

Logically, the converse is not equivalent to the original statement, so it is reasonable that only one of the statements is true. However, stating that the two are not logically equivalent is not a sufficient proof that the converse is false. This is because the two statements do not contradict; rather, if both held, they would make a stronger "if and only if" condition.