CS/Math 240: Intro to Discrete Math 2/8/2011

Solutions to Homework 1

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Problem 1

We will define each of the supplied sentences using propositional logic, using propositional variables with the following meaning:

- A: a divides bc
- B: a divides b
- C: a divides c

So our original statement reads $A \Rightarrow (B \vee C)$.

(a) $(B \vee C) \Rightarrow A$; this is the converse of the original statement. It is not equivalent to the original statement: an example of a truth assignment where the two do not agree is A = True, B = True, and C = False.

(b) $(\neg B \vee \neg C) \Rightarrow \neg A$. This appears similar to the contrapositive, but it is actually neither the contrapositive nor the converse. This can be verified, as the statement evaluates to False on the assignment A = True, B = False, C = True, whereas the initial proposition, its contrapositive, and its converse evaluate to True.

(c) $A \wedge \neg B \wedge \neg C$. This is neither the contrapositive nor the converse of the original statement. It is actually true only in the case where the original statement is false; hence the two statements are obviously not equivalent.

(d) $(\neg B \wedge \neg C) \Rightarrow \neg A$. This is the contrapositive of the original statement; it can be seen by DeMorgan's Law, $\neg(B \vee C) \iff \neg B \wedge \neg C$. This statement is equivalent to the original statement, which can be shown either by definition of the contrapositive or by a full enumeration of the truth tables for each statement.

(e) $(A \wedge \neg C) \Rightarrow B$. This is not the contrapositive nor the converse of the original statement; however, it is equivalent to the original statement. This can be shown by a full enumeration of the truth tables for each statement. Alternately, it can be seen that both statements are false only when their antecedent is True and the consequence False, which in both cases is the single assignment A = True, B = False, and C = False.

We can alternately argue equivalence by rewriting statement (e) in a number of steps; knowing that $(P \Rightarrow Q) \iff (\neg P \vee Q)$ for any two propositions P and Q, we see that $((A \wedge \neg C) \Rightarrow B) \iff (\neg(A \wedge \neg C) \vee B)$, which after applying DeMorgan's Law becomes $(\neg A \vee C) \vee B$. At this point, we can rearrange our parentheses, since 'or' operations can be performed in any order. We arrive at $\neg A \vee (B \vee C)$, which translates to our original statement $A \Rightarrow (B \vee C)$.

Problem 2

Part a

A more compact way of writing P_1: $P \wedge (Q \vee P)$ is P. We first show that by comparing the truth tables of P and $P \wedge (Q \vee P)$. See Table 1.

P	Q	$Q \vee P$	$P \wedge (Q \vee P)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Table 1: Truth table for problem 2, part a. The first and the last column prove that P and $P \wedge (Q \vee P)$ are logically equivalent.

We can also argue logical equivalence without the use of truth tables. Observe that (i) when P is true, $Q \vee P$ is also true, and (ii) if P is false, then no matter how we set Q, $P \wedge (Q \vee P)$ is going to be false. Thus, P_1 is true iff P is.

Since the truth of P_1 depends on P, we cannot use a smaller number of variables than 1, which means that our solution is minimal.

Part b

Here we work with the proposition P_2: $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$. We'll see that we can rewrite it with as $\neg P \vee \neg Q \vee R$.

Let's start by writing down the truth table for P_2 as Table 2. We see that the only way for P_2 to be false is when P and Q are true, and R is false. So P_2 is false iff $P \wedge Q \wedge \neg R$, which means it's true iff $\neg(P \wedge Q \wedge \neg R)$. This is logically equivalent to $\neg P \vee \neg Q \vee R$ by DeMorgan's law.

P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q \vee R$
T	T	T	T	T	T	F	T	T
T	T	F	T	F	F	F	T	F
T	F	T	F	T	T	F	F	T
T	F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	T	T	T	F	T
F	F	F	T	T	T	T	F	T

Table 2: Truth table for $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ and $\neg P \vee \neg Q \vee R$.

We now present an argument that does not require truth tables. Observe that P_2 can only be false if $P \Rightarrow Q$ holds and $P \Rightarrow R$ is false. The latter happens if and only if P is true and R is false. In that case, $P \Rightarrow Q$ holds if and only if Q holds. Therefore, P_2 is false if and only if $P \wedge Q \wedge \neg R$. As we argued before, this means it's true if and only if $\neg P \vee \neg Q \vee R$ by DeMorgan's law.

To argue that we cannot use fewer variables, notice that each variable influences the truth of P_2 (because changing the value of any variable in the setting $P =$ true, $Q =$ true, $R =$ false changes the truth value of P_2), so each variable must occur in the expression for P_2 at least once. In our solution, each variable occurs exactly once, which means that our solution uses the smallest number of variable occurrences possible.

Part c

Now we work with the proposition P_3: $(P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R)$. Proposition P_3 is true if and only if either all or two of the three variables are true. This expresses the idea of *majority*.

See the truth table in Table 3 which shows that the majority function is logically equivalent to $(P \wedge (Q \vee R)) \vee (Q \wedge R)$.

P	Q	R	majority	$Q \vee R$	$P \wedge (Q \vee R)$	$Q \wedge R$	$(P \wedge (Q \vee R)) \vee (Q \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	T	F	T
T	F	F	F	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	F	T	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Table 3: Truth table for the majority function.

Again, let's give an argument without truth tables. Proposition P_3 is true if the majority of the variables are true, and is false otherwise. This means that we need either P and Q to hold, or P and R to hold, or Q and R to hold, so an alternative way of writing P_3 is $(P \wedge Q) \vee (P \wedge R) \vee (Q \wedge R)$.

We can further reduce the number of variable occurrences by applying the distributive law which says that $(P \wedge Q) \vee (P \wedge R)$ is logically equivalent to $P \wedge (Q \vee R)$. Therefore, we rewrite P_3 as $(P \wedge (Q \vee R)) \vee (Q \wedge R)$.

Our final expression for the majority has a fairly intuitive explanation. There are two cases to consider, and if any of them is true, then the majority function evaluates to true, which means that there is an "or" between the two cases. If P is true, we need at least one of Q and R to be true. In the other case, we assume P is false, and we need both Q and R to be true.

Problem 3

Let's try to interpret the given propositions. P_1 makes use of two quantifiers. Let us ignore $(\forall i \in \mathbb{N})$ for the time being. The rest of the expression can be read as – for a given i, there exists a j such that $j > i$ and $a_j > a_i$ or $a_j < a_i$. In other words, for a given term a_i in the sequence, another term a_j exists such that a_j appears after a_i and a_j is different from a_i. Now include $(\forall i \in \mathbb{N})$, then the complete proposition can be read as – for each term a_i in the sequence, there is another term a_j that appears after a_i in the sequence, and is different from a_i.

Similarly we can interpret proposition P_2 as – for each term a_i in the sequence, a term exists that appears after a_i in the sequence and is greater than a_i, and another term exists which also appears after a_i but is less than a_i. As a fact of interest, note that $P_2 \Rightarrow P_1$, but the converse is not true in general.

(a) First, let us try to figure out a pattern for the given sequence. Take look at the terms indexed by multiples of 3 i.e. the terms $a_0,\ a_3,\ \ldots$, these together are consecutive terms in the sequence of natural numbers starting from 1. Thus the general for the term a_{3k} would be $k+1$ where $k \in \mathbb{N}$. Similarly, the general expressions for terms a_{3k+1} and a_{3k+2} are $k+2$ and $k+3$ respectively.

    With this understanding of the sequence, we can easily check whether or not the given propositions hold true for the sequence. P_1 holds for the sequence as for each i, we can always find a j such that $j > i$ and $a_j \neq a_i$. For example, one can check that a_{i+3} is always different from a_i. But P_2 does not hold for the sequence. This is because there is no term that appears after a_0 in the sequence and is less than a_0.

(b) In this sequence we can observe that initially natural numbers appear from 10 to 1 in the decreasing order. After that, the natural number 1 repeats forever. Again, it is easy to prove that neither P_1 nor P_2 hold true for this sequence. This is because once 1 starts repeating, no term different from 1 can appear in the sequence. For all terms a_i such that $i \geq 10$, both P_1 and P_2 fail to hold.

Problem 4

(a) The given sentence is of the form – If P, then Q. where

$$P: \text{ All students attended the meeting}$$
$$Q: \text{ All faculty members attended the meeting}$$

    We can symbolically express the sentence as – $P \Rightarrow Q$.
    Now, P can be read as –

    For each x, if x is a student, then x attended the meeting.

    Symbolically, this may be expressed as – $(\forall x \in D)(S(x) \Rightarrow A(x))$

    One may commit a mistake in writing P as – $(\forall x \in D)(S(x) \wedge A(x))$. In fact one of the TAs did manage to do exactly that. This is incorrect because this representation implies that – for each x, x is a student and x attended the meeting. In other words, everyone is a student and everyone attended the meeting.

    Similarly, Q's symbolic representation would be – $(\forall x \in D)(F(x) \Rightarrow A(x))$
    Combining these together, we get –

$$(\forall x \in D)(S(x) \Rightarrow A(x)) \Rightarrow (\forall x \in D)(F(x) \Rightarrow A(x))$$

(b) Expressing the given proposition in words, we have – there is a person x and a person y such that x is student and y is a parent of x and y did not attend the meeting.

Simplifying this further, we have – there is a student and a parent of the student s.t. the parent did not attend the meeting.

Finally, one may write it as – some student's parent did not attend the meeting.

(c) The first thing to take notice is that 'some' appears at the beginning of the sentence. This suggests the use of the existential quantifier – $\exists$. So one may transform the sentence to – there exists a student x such that x knows everyone who attended the meeting.

The sentence can be further re-written as – there exists a student x such that everyone who attended the meeting is known to x.

'Everyone' suggests that we make use of the universal quantifier – $\forall$. Since x is already in use, we need a new variable. We re-write the sentence – there exists a student x such that, for each person y, if y attended the meeting, then x knows y.

The final symbolic representation would be – $(\exists x)(S(x) \wedge ((\forall y)(A(y) \Rightarrow K(x, y))))$

(d) The presence of $\Rightarrow$ suggests that the part of the sentence after the universal quantification has the structure – if P, then Q. We can transform the given representation to – for each x, if x is a student and Bob knows x, then x did not attend the meeting.

This can be rewritten as – none of the students known to Bob attended the meeting. Another way of expressing could be – Bob does not know any of the students who attended the meeting.

(e) The given sentence can be re-phrased as – there does not exist x such that x is a student and x attended the meeting and x knows some faculty member.

We can further re-write the sentence as – there does not exist x such that x is a student and x attended the meeting and some faculty member is known to x.

'not' suggests use of the negation operator, 'some' suggests use of the existential quantifier. So the final form would be – $\neg(\exists x)(S(x) \wedge A(x) \wedge ((\exists y)(F(y) \wedge K(x, y))))$

(f) The first step would be to express the proposition after the univesal quantification using the - if P, then Q structure. That would yield – for all x, if x is a faculty member and x attended the meeting, then Bob knows x.

This can be further re-written as – if a faculty member attended the meeting, then Bob knows that faculty member.

Finally, we get to the form – Bob knows every faculty member who attended the meeting.

(g) The first part of the sentence is straightforward to represent using symbols: $A(\text{Alice}) \wedge A(\text{Bob})$. "But" is logically equivalent to "and". The last part of the sentence can be rephrased as "Alice does not know any student and Bob does not know any student". The former is equivalent to saying that everyone who is a student is not known by Alice, i.e., $(\forall x)(S(x) \Rightarrow \neg K(\text{Alice}, x))$. Replacing "Alice" by "Bob" gives the translation of the latter part. We can merge the quantifications to obtain our final answer: $A(\text{Alice}) \wedge A(\text{Bob}) \wedge (\forall x)\,(S(x) \Rightarrow \neg K(\text{Alice}, x) \wedge \neg K(\text{Bob}, x))$

(h) We can re-write the given proposition as – for all x and y, if there exist z and w such that z and w are students and z and w attended the meeting and x is a parent of z and y is parent of w, then x and y know each other.

This may be shortened to – for all x and y, if there exist students z and w who attended the meeting and x and y are their parents respectively, then x and y know each other.

This can be further simplified to – parents of students who attended the meeting, know each other.

(i) This statement is of the form "P unless Q", where where

$$P: \text{ Alice will attend the meeting}$$
$$Q: \text{ Alice knows no student who will attend the meeting.}$$

The phrase "P unless Q" means "If P does not hold then Q has to hold", i.e., $\neg P \Rightarrow Q$, or equivalently, "If Q does not hold then P holds", i.e., $\neg Q \Rightarrow P$. Using the latter, the given sentence can be paraphrased as – Alice will attend the meeting if she knows some student who will attend the meeting. As we can express P as $A(\text{Alice})$, and Q as $\neg(\exists x)(S(x) \wedge A(x) \wedge K(\text{Alice}, x))$, we end up with $((\exists x)(S(x) \wedge A(x) \wedge K(\text{Alice}, x))) \Rightarrow A(\text{Alice})$

(j) The proposition can be re-written as – there exists a person x such that x is faculty member and x attended the meeting and x knows Alice and x knows the parents of Alice.

This can be simplified as – some faculty member attended the meeting and knows Alice and her parents.

Problem 5

We will begin by introducing four propositional variables with the following intended meaning:

- A: Alice is innocent
- B: Bob is innocent
- C: Carol is innocent
- D: David is innocent

Then, our sentences can be translated in order as the following:

P_{Alice}: $(\neg B \wedge C) \vee (B \wedge \neg C)$

P_{Bob}: $(A \vee D) \Rightarrow C$

P_{Carol}: $(\neg A \vee \neg B) \wedge C$

P_{David}: $(B \vee \neg C) \iff (B \wedge \neg C)$

It is worth noting the ambiguity of both Alice and Carol's statements; the 'or' in each of their statements could potentially be either inclusive or exclusive. For Alice's statement, $(\neg B \wedge C)$ and $(B \wedge \neg C)$ are by definition mutually exclusive, so the type of 'or' is not important; however, for Carol's statement using an inclusive or exclusive or could potentially affect the values of our truth assignments. In this case, either interpretation will lead to the same resolution, and either is acceptable as part of a submitted solution.

A second immediate note that simplifies our problem is the fact that P_{Alice} and P_{David} are logically equivalent statements; furthermore, both of these statements can be minimalized to the form $B \iff \neg C$. Hence, for part (a) we can ignore P_{David}, as satisfying P_{Alice} will naturally lead to P_{David} being satisfied. We will also use the minimization of P_{Alice} in part (b) to simplify our reasoning.

Part a

It is possible to create a large truth table to enumerate all possible variable assignments to check for the existence of an assignment that resolves each statement to True. However, such an assignment can also be found in a more clever manner. Note that for Carol to be telling the truth, she must be innocent; so C = True must be part of the final solution. In this case, P_{Bob} is now necessarily true. With C = True, P_{Alice} can only be satisfied if B = False; at this point, P_{Carol} is also satisfied if we assume an inclusive or. Even if the or in P_{Carol} is exclusive, setting A = True will satisfy the statement. So now we have a satisfying assignment A = True, B = False, C = True, where D can be either True or False.

Part b

Note that for each person's statement, that statement is now true if and only if they are innocent. This can be expressed with our existing propositions as such (remembering to minimize the applicable sections of P_{Alice} and P_{David}):

P_{Alice}: $A \iff (B \iff \neg C)$

P_{Bob}: $B \iff ((A \vee D) \Rightarrow C)$

P_{Carol}: $C \iff ((\neg A \vee \neg B) \wedge C)$

P_{David}: $D \iff (B \iff \neg C)$

The question at hand is whether it is possible to determine who is guilty and who is innocent. In order for this to be the case, not only must there be a valid assignment for all propositions, but there must be only one such assignment (otherwise it would not be possible to distinguish between the two). Again, for this problem it is satisfactory to enumerate the truth table for each statement and determine where, if at all, each of the statements is simultaneously satisfied. This process is sped up somewhat since we can ignore any assignment that evaluates to False for a given statement when evaluating subsequent statements.

However, a more clever method to solving the problem is to first note that, while P_{Alice} and P_{David} are no longer logically equivalent, the only assignments for which these statements do not create a contradiction are $A \iff D$. Given this information, we can replace any occurrence of the variable D with A, resulting in the following three statements (after the replacement, we will again be able to discard P_{David}):

P'_{Alice}: $A \iff (B \iff \neg C)$

P'_{Bob}: $B \iff (A \Rightarrow C)$

P'_{Carol}: $C \iff ((\neg A \vee \neg B) \wedge C)$

Now, we will examine possible variable assignments by 'branching' on C; that is, we will explore both the case where C = True and C = False.

Case One (C = True). Examining P'_{Bob}, we see that $A \Rightarrow C$ evaluates to True. As such, we must have B = True. Then, from P'_{Alice}, we must also have A = False. This set of assignments will satisfy P'_{Carol} regardless of whether the 'or' in the statement is exclusive or inclusive. So we have found a satisfying assignment, and since the values of A, B, and D directly followed from C = True, this is the only satisfying assignment in the case that C = True.

Case Two (C = False). We see that with C = False, the right-hand side of P'_{Alice} holds iff B is True, so P'_{Alice} simplifies to $A \iff B$. Similarly, with C = False, the right-hand side of P'_{Bob} holds iff A is False, so P'_{Bob} simplifies to $B \iff \neg A$. We cannot have both $A \iff B$ and $B \iff \neg A$, so there is no possible satisfying assignment with C = False.

Thus, after examining all possibilities the only legal assignment to satisfy all statements is A = False, B= True, C = True, and D = False. So it is possible to determine who is guilty and who is innocent, by the method we just employed.